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Phase diagram of inhomogeneous percolation with a defect plane

Received: date / Accepted: date

Abstract Let \mathbb{L} be the d -dimensional hypercubic lattice and let \mathbb{L}_0 be an s -dimensional sublattice, with $2 \leq s < d$. We consider a model of inhomogeneous bond percolation on \mathbb{L} at densities p and σ , in which edges in $\mathbb{L} \setminus \mathbb{L}_0$ are open with probability p , and edges in \mathbb{L}_0 open with probability σ . We generalize several classical results of (homogeneous) bond percolation to this inhomogeneous model. The phase diagram of the model is presented, and it is shown that there is a subcritical regime for $\sigma < \sigma^*(p)$ and $p < p_c(d)$ (where $p_c(d)$ is the critical probability for homogeneous percolation in \mathbb{L}), a bulk supercritical regime for $p > p_c(d)$, and a surface supercritical regime for $p < p_c(d)$ and $\sigma > \sigma^*(p)$. We show that $\sigma^*(p)$ is a strictly decreasing function for $p \in [0, p_c(d)]$, with a jump discontinuity at $p_c(d)$. We extend the Aizenman-Barsky differential inequalities for homogeneous percolation to the inhomogeneous model and use them to prove that the susceptibility is finite inside the subcritical phase. We prove that the cluster size distribution decays exponentially in the subcritical phase, and sub-exponentially in the supercritical phases. For a model of lattice animals with a defect plane, the free energy is related to functions of the inhomogeneous percolation model, and we show how the percolation transition implies a non-analyticity in the free energy of the animal model. Finally, we present simulation estimates of the critical curve $\sigma^*(p)$.

Keywords Percolation · Phase diagram · Inhomogeneous percolation

PACS 64.60.Ak; 64.80.Gd; 05.20.-y; 02.50.+s

Mathematics Subject Classification (2000) 82B43 ; 60K35

Research supported by grants from NSERC (Canada)

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1 Introduction

Percolation [5] in \mathbb{Z}^d is a lattice model of polymeric gelation [35,37] and of chemical gelation due to polymerisation of monomers or comonomers [16]. In a percolation model the phenomenon of gelation is understood as a critical phenomenon [7] with characteristic scaling about a critical point called the percolation threshold. Studies of gelation from a percolation point of view are now classical [38,41,26], and were reviewed in references [39,13,21,40,14].

Surface phenomena in percolation have also received considerable attention [6,8,9,10,11,12]. This is a model of gelation along a defect plane or surface, and was also interpreted as a model of branch polymer adsorption in bulk [11] – results in references [8,11] suggest that a surface transition is absent in two dimensional models.

Consider a model of inhomogeneous percolation in the hypercubic lattice \mathbb{Z}^d with an s -dimensional hyperplane \mathbb{Z}^s as a defect plane (where $2 \leq s < d$). This model has received some attention in the mathematical literature [33] (see reference [29] for a model of inhomogeneous percolation with defect lines in the bulk lattice).

Percolation along a defect plane may be considered as a model of gelation along a surface defect, and it is known that this phenomenon is associated with a *surface transition* in addition to the usual bulk percolation phenomenon [10,9,11,33].

There is a significant number of results known for (homogeneous) bond percolation [17,25]. Known results include the location of the critical bond-percolation threshold in the square lattice [24] (see also [22]), the uniqueness of the critical point [1,31] and the decay rate of the clusters at the origin in the sub- and supercritical phases [2,3]. Analogous results for models of inhomogeneous percolation are incomplete, and in this paper our aim is to provide some mathematical results to extend the standard theorems of homogeneous percolation to a model of inhomogeneous percolation. This requires generalisation of several of the classical results for homogeneous bond percolation. A secondary goal is examine the connection between lattice models of branch polymers close to a surface or defect plane and percolation along a defect plane.

1.1 Homogeneous percolation

In this section we define some terms and notation, and we briefly review homogeneous percolation.

The d -dimensional hypercubic lattice \mathbb{L} with vertices in \mathbb{Z}^d has unit length edges joining nearest neighbour vertices (or points) in \mathbb{Z}^d . The set of edges of \mathbb{L} is denoted by \mathbb{E} . We shall write $x \sim y$ to denote the edge that joins the vertices x and y .

In bond percolation models, each edge $e \in \mathbb{E}$ has an associated random variable $\omega(e)$ with possible values 0 and 1. We say that the edge e is *open* if $\omega(e) = 1$, and that e is *closed* if $\omega(e) = 0$. In the present paper we always assume that the random variables $\omega(e)$ are independent.

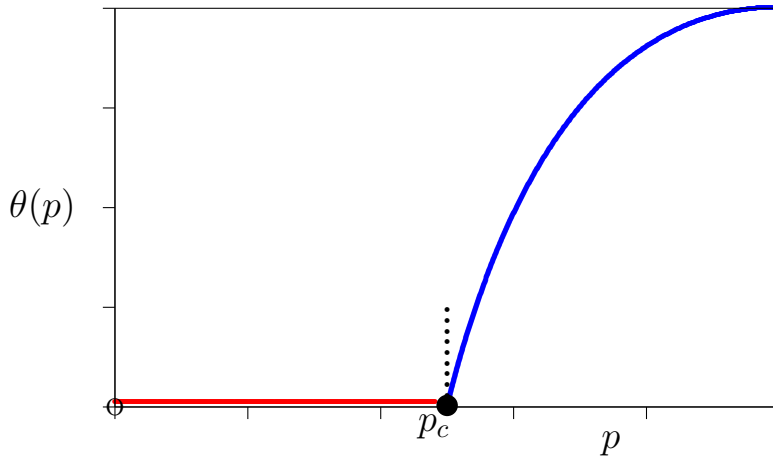


Fig. 1 A schematic graph of the probability that the cluster at the origin is infinite as a function of p in homogeneous bond percolation. This probability is zero for $p < p_c(d)$ and positive for $p > p_c(d)$.

In the *homogeneous* percolation model, the probability that $\omega(e) = 1$ is the same for every e , and we denote this common value by p .

We call p the *density* of the model. We denote by P_p^H the homogeneous (bond)-percolation measure on \mathbb{L} at density p , and by E_p^H the expectation with respect to P_p^H . (The superscript “H” will be used for functions describing homogeneous percolation, in contrast to the inhomogeneous model to be introduced below.)

The union of open edges is a subgraph G of \mathbb{L} . In general, G is not a connected graph, but is the union of a collection of connected subgraphs of open edges. For a vertex x , let $C(x)$ be the connected component of open edges containing x . We call $C(x)$ the open cluster at x .

When x is the origin, we write C instead of $C(0)$.

The collection of closed edges incident with $C(x)$ is the *perimeter* of $C(x)$.

The *size* of the cluster C is the number of vertices in C , and is denoted $|C|$. We shall also work with the number of edges in C , which we denote by $\|C\|$.

The probability that the origin is in a cluster of infinite size is given by

$$\theta_d^H(p) \equiv \theta^H(p) = \lim_{n \rightarrow \infty} P_p^H(|C| \geq n) = P_p^H(|C| = \infty) \quad (1)$$

(we shall often omit the subscript d).

The fundamental property of percolation is that there exists a critical density $p_c(d) \in (0, 1]$ in the d -dimensional lattice (see reference [17], section 1.4) such that

$$\theta_d^H(p) \quad \begin{cases} = 0 & \text{if } p < p_c(d), \\ > 0 & \text{if } p > p_c(d). \end{cases} \quad (2)$$

It is easy to see that $p_c(1) = 1$; however the result that $p_c(2) = \frac{1}{2}$ requires considerably more effort [24]. In general, it can be shown that $0 < p_c(d+1) < p_c(d)$ [25, 32].

The expected value of the size of the cluster at the origin is the *susceptibility* defined by

$$\chi^H(p) = E_p^H|C| = [\infty \cdot P_p^H(|C|=\infty)] + \sum_{n=1}^{\infty} n P_p^H(|C|=n) \quad (3)$$

(interpreting $\infty \cdot 0 = 0$).

If $p > p_c(d)$, then obviously $\chi^H(p) = \infty$. It is also known that $\chi^H(p) < \infty$ whenever $p < p_c(d)$ (see references [1,31], and also for example [17]). This property is often referred to as the uniqueness of the critical point. The finite component of the susceptibility is given by

$$\chi^{f,H}(p) = E_p^H(|C| \mathbb{1}_{|C|<\infty}) = \sum_{n=1}^{\infty} n P_p^H(|C|=n). \quad (4)$$

Clearly $\chi^{f,H}(p) \leq \chi^H(p)$.

It is known that the limit

$$\zeta^H(p) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log P_p^H(|C|=n) \quad (5)$$

exists and that $\zeta^H(p) > 0$ if $p < p_c(d)$ [27]. Hence, we have exponential decay of $P_p^H(|C|=n)$ in the subcritical regime. More explicitly, $P_p^H(|C|=n)$ is bounded from above by [3]

$$P_p^H(|C|=n) \leq P_p^H(|C|\geq n) \leq 2 e^{-\frac{n}{2[\chi^H(p)]^2}} \quad \text{for all } n, \text{ if } p < p_c(d). \quad (6)$$

In the supercritical phase the cluster size distribution of the cluster at the origin has sub-exponential decay [2]:

$$P_p^H(|C|=n) \geq e^{-\gamma(p) n^{(d-1)/d}} \quad (7)$$

where $\gamma(p)$ is a finite function of $p \in (p_c(d), 1]$. By taking logarithms, dividing by n and letting $n \rightarrow \infty$, this shows that $\zeta^H(p) = 0$ for $p > p_c(d)$.

1.2 Inhomogeneous percolation

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}$ be the standard basis of unit vectors in the d -dimensional hypercubic lattice \mathbb{L} . Choose an integer s such that $2 \leq s < d$ and let \mathbb{L}_0 be the s -dimensional sublattice of \mathbb{L} which contains the origin and has basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_s\}$.

We shall view \mathbb{L}_0 as a “defect plane” or “adsorbing surface” in \mathbb{L} . The set of edges or bonds with both endpoints in \mathbb{L}_0 is \mathbb{E}_0 , and we shall write that $\mathbb{L}_0 \subseteq \mathbb{L}$, since $\mathbb{E}_0 \subseteq \mathbb{E}$.

Inhomogeneous bond percolation is set up in \mathbb{L} with one density σ for the defect plane \mathbb{E}_0 and another density p for the bulk $(\mathbb{E} \setminus \mathbb{E}_0)$. Given $p, \sigma \in [0, 1]$, the inhomogeneous percolation probability measure $P_{p,\sigma}^I$ is given by

$$P_{p,\sigma}^I(\omega(e)=1) = P_{p,\sigma}^I(e \text{ is open}) = \begin{cases} p & \text{if } e \in \mathbb{E} \setminus \mathbb{E}_0, \\ \sigma & \text{if } e \in \mathbb{E}_0, \end{cases} \quad (8)$$

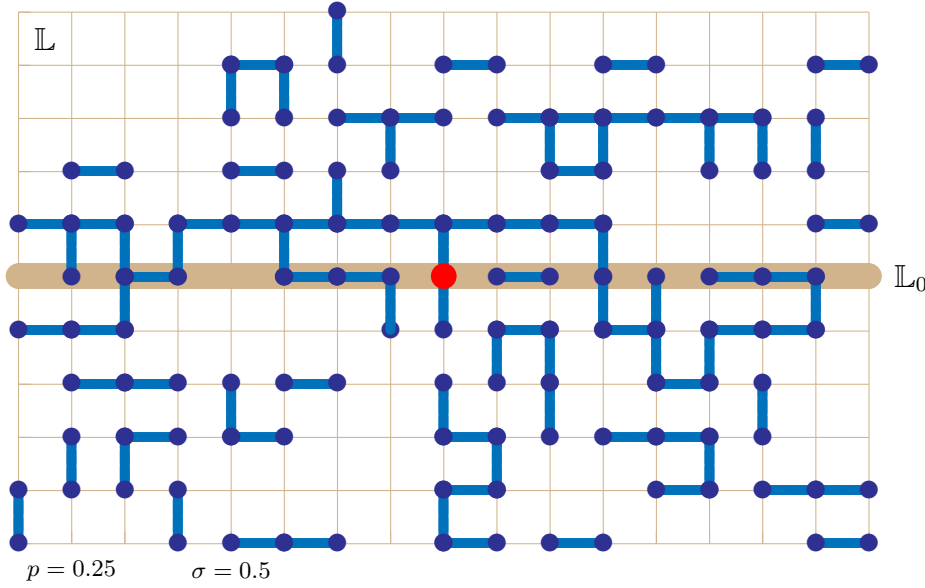


Fig. 2 Bond percolation in the square lattice \mathbb{L} with a (one dimensional) defect line \mathbb{L}_0 . Edges in \mathbb{L}_0 are open with probability σ , and (bulk) edges in $\mathbb{L} \setminus \mathbb{L}_0$ are open with density p . In this illustration a section of \mathbb{L} centered at the origin is shown and the densities were set at $p = 0.25$ and $\sigma = 0.5$.

with all edges independent. The corresponding expectation is $E_{p,\sigma}^I$.

Open clusters $C(x)$ are defined as before. The probability that C (the cluster at the origin) has infinite size is given by

$$\theta^I(p, \sigma) = \lim_{n \rightarrow \infty} P_{p,\sigma}^I(|C| \geq n) = P_{p,\sigma}^I(|C| = \infty) \quad (9)$$

(we suppress the dimensions d and s in this notation). We say that percolation occurs if $\theta^I(p, \sigma) > 0$. Clearly, $\theta^I(p, p) = \theta^H(p)$, and $\theta^I(p, \sigma)$ is a non-decreasing function of its arguments—that is, $\theta^I(p, \sigma) \leq \theta^I(p', \sigma')$ if $p \leq p'$ and $\sigma \leq \sigma'$.

Similarly, the susceptibility is defined by

$$\chi^I(p, \sigma) = E_{p,\sigma}^I|C| = [\infty \cdot P_{p,\sigma}^I(|C| = \infty)] + \sum_{n=1}^{\infty} n P_{p,\sigma}^I(|C| = n) \quad (10)$$

and we define

$$\chi^{f,I}(p, \sigma) = E_{p,\sigma}^I(|C| \mathbb{1}_{|C| < \infty}) = \sum_{n=1}^{\infty} n P_{p,\sigma}^I(|C| = n). \quad (11)$$

Figure 3 shows the three regimes of this model. We shall begin with a formal definition of the three regimes \mathcal{R}_0 , \mathcal{R}_L and \mathcal{R}_H , and then we shall describe their properties.

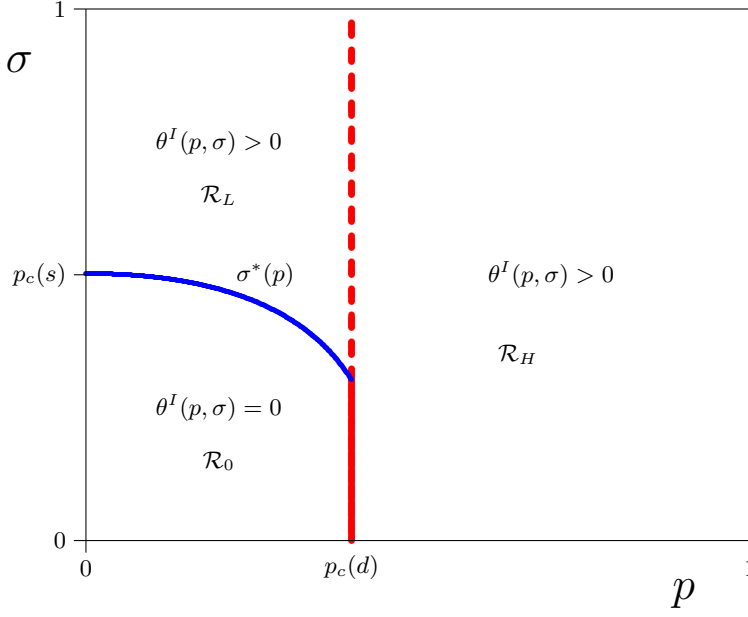


Fig. 3 The phase diagram of inhomogeneous percolation. The subcritical phase is labeled by \mathcal{R}_0 , and we distinguish two supercritical phases: The regime labeled by \mathcal{R}_L is a surface supercritical phase, where percolation has occurred along \mathbb{L}_0 but has not penetrated into the bulk of the lattice \mathbb{L} . In the regime marked by \mathcal{R}_H percolation occurs throughout the d -dimensional lattice \mathbb{L} , since $p > p_c(d)$.

We define a critical curve $\sigma = \sigma^*(p)$ by

$$\sigma^*(p) = \inf\{\sigma \in [0, 1] \mid \theta^I(p, \sigma) > 0\} \quad \text{for } p \in [0, 1]. \quad (12)$$

It is not hard to show that $\sigma^*(p) = 0$ for $p > p_c(d)$, and $0 < \sigma^*(p) < 1$ for $0 \leq p < p_c(d)$ (see proposition 1). Accordingly, we define

$$\begin{aligned} \mathcal{R}_0 &= \{(p, \sigma) \in [0, 1]^2 : p < p_c(d) \text{ and } \sigma < \sigma^*(p)\}, \\ \mathcal{R}_L &= \{(p, \sigma) \in [0, 1]^2 : p < p_c(d) \text{ and } \sigma > \sigma^*(p)\}, \text{ and} \\ \mathcal{R}_H &= \{(p, \sigma) \in [0, 1]^2 : p > p_c(d)\}. \end{aligned}$$

By definition, we see that $\theta^I(p, \sigma) = 0$ at every point of \mathcal{R}_0 . Thus \mathcal{R}_0 is the subcritical phase, in which every cluster is finite. Also by the definition (12), we have

$$\sigma^*(0) = p_c(s). \quad (13)$$

In \mathcal{R}_H , the infinite cluster extends throughout the bulk. Indeed, for given (p, σ) in \mathcal{R}_H , the probability $P_{p, \sigma}^I(\mathbf{v} \in C)$ is bounded away from 0 uniformly for all \mathbf{v} in \mathbb{L} (see proposition 3). In contrast, for (p, σ) in \mathcal{R}_L , the probability $P_{p, \sigma}^I(v \in C)$ decays to 0 exponentially rapidly in the distance from v to \mathbb{L}_0 (see lemma 2). Thus we call \mathcal{R}_L the surface supercritical phase, where the infinite cluster stays close to the defect plane rather than penetrating into the bulk, and we call \mathcal{R}_H the bulk supercritical phase, where the infinite

cluster spreads throughout the whole lattice. Alternatively, for large N we see that $E(|C| \cap [-N, N]^d)$ is proportional to N^d when (p, σ) is in \mathcal{R}_H , and is proportional to N^s when (p, σ) is in \mathcal{R}_L .

From a slightly different perspective, if we fix $p < p_c(d)$, then increasing σ takes the model through a percolation transition at $\sigma^*(p)$ from \mathcal{R}_0 into \mathcal{R}_L . This transition is often referred to as a “surface phase transition” in the model ([12], see references [10, 9] as well).

(We remark here that the case $s = 1$, a defect line, has simpler behaviour: for every σ , we have $\theta^I(p, \sigma) = 0$ if $p < p_c(d)$ and $\theta^I(p, \sigma) > 0$ if $p > p_c(d)$ [29]. That is, there is no \mathcal{R}_L phase. For this reason, we assume $s \geq 2$ throughout this paper.)

If p is increased so that $p > p_c(d)$ then the model goes through a bulk percolation threshold into a bulk super-critical regime \mathcal{R}_H — see proposition 1.

The boundary between the regimes \mathcal{R}_0 and \mathcal{R}_L will be denoted by $\mathcal{R}_{0,L}$. Obviously, the critical curve $\sigma = \sigma^*(p)$ for $p \in [0, p_c(d))$ is a subset of $\mathcal{R}_{0,L}$. We expect that they are equal (except perhaps for a point at $p = p_c(d)$), but we do not know this rigorously because we cannot prove that the curve is continuous on $[0, p_c(d))$. The curve is obviously non-increasing, and Proposition 2 shows that it is strictly decreasing on $[0, p_c(d)]$.

There is a jump discontinuity in $\sigma^*(p)$ at $p_c(d)$. Indeed, proposition 1 in section 2 shows that $p_c(s) > \sigma^*(p) > p_c(d)$ for every p in $(0, p_c(d))$ and that $\sigma^*(p) = 0$ if $p > p_c(d)$. Thus, the boundary between \mathcal{R}_0 and \mathcal{R}_H is a vertical line segment at $p = p_c(d)$. For large enough d , Newman and Wu [33] prove the stronger result that $p_c(s) > \sigma^*(p_c(d)) > p_c(d)$ for $2 \leq s \leq d - 3$. They conjecture that this is true whenever $2 \leq s < d$. We prove that $p_c(s) > \sigma^*(p_c(d))$ in general (see corollary 1). It would be much harder to prove $\sigma^*(p_c(d)) > p_c(d)$, since this would imply the longstanding conjecture that $\theta_d^H(p_c(d)) = 0$. In general, it seems hard to say much about $\sigma^*(p_c(d))$.

In section 3 we consider the uniqueness of the critical point. This requires the generalisation of differential inequalities for homogeneous percolation in reference [1] to the model in this paper. This is done in A, and the resulting modified inequalities are used to show that if $\chi^I(p, \sigma) = \infty$, then (p, σ) cannot be in the interior of \mathcal{R}_0 (see theorem 1).

In sections 4 and 5, we consider the distribution of the size of the cluster C at the origin. In the subcritical regime \mathcal{R}_0 , Theorem 3 shows that the distribution of $|C|$ has exponential tails; more precisely, for every n

$$P_{p,\sigma}^I(|C|=n) \leq 2 e^{-n/(2\chi^H(p)\chi^I(p,\sigma))}. \quad (14)$$

In the supercritical regime \mathcal{R}_H , there exists a $\gamma(p) > 0$ such that

$$P_{p,\sigma}^I(\infty > |C| \geq n) \geq P_{p,\sigma}^I(|C|=n) \geq e^{-\gamma(p)n^{(d-1)/d}}. \quad (15)$$

See theorem 4 in section 5. This result should be compared with the situation in regime \mathcal{R}_L , where we show in theorem 5 that there exist positive β_1 and β_2 (depending on p and σ) such that

$$P_{p,\sigma}^I(\infty > |C| \geq n) \geq \beta_1 e^{-\beta_2 n^{(s-1)/s} (\log^2 n)^{d-s}}. \quad (16)$$

It follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{(d-1)/d}} \log P_{p,\sigma}^I(\infty > |C| \geq n) = 0 \quad \text{in regime } \mathcal{R}_L. \quad (17)$$

This suggests two different behaviours for the tails of $P_{p,\sigma}^I(\infty > |C| \geq n)$ in the regimes \mathcal{R}_L and \mathcal{R}_H .

In section 6 we consider briefly the relation of $P_{p,\sigma}^I(|C| = n)$ to a lattice animal model of polymer collapse near a defect plane. We show that there is a limiting free energy for the lattice animals which implies the existence of the limits

$$\psi^I(p, \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log P_{p,\sigma}^I(|C| = n), \quad \zeta^I(p, \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log P_{p,\sigma}^I(|C| = n). \quad (18)$$

Using our knowledge of the percolation transition, we show that the lattice animal free energy is non-analytic on certain curves.

We coded the numerical algorithm of Newman and Ziff [34] for the inhomogeneous percolation model and collected data to determine the location of the critical curve $\sigma^*(p)$ for low dimensions. We present some results in section 7, including data on the case with the bulk density fixed at density p near $p_c(d)$, where we obtain estimates of the curve σ^* , consistent with reference [6].

We conclude the paper in section 8 with a summary and some final remarks on the model.

2 The Phase Boundaries

This section proves properties of the critical curve $\sigma = \sigma^*(p)$, which is defined in equation (12) by

$$\sigma^*(p) = \inf\{\sigma \in [0, 1] \mid \theta^I(p, \sigma) > 0\}.$$

Let $P_{p,\sigma}^I$ and $\theta^I(p, \sigma)$ be defined as in equations (8) and (9), with the homogeneous analogues P_p^H and $\theta^H(p)$ as defined in section 1.1. Observe that $P_{p,p}^I = P_p^H$, $\theta^I(p, p) = \theta^H(p) \equiv \theta_d^H(p)$ and $\theta^I(0, \sigma) = \theta_s^H(\sigma)$.

The following proposition verifies the basic structure of Figure 3.

Proposition 1 *The critical curve $\sigma = \sigma^*(p)$ satisfies*

- (a) $\sigma^*(0) = 0$ for $p > p_c(d)$, and
- (b) $0 < p_c(d) \leq \sigma^*(p) \leq p_c(s) = \sigma^*(0)$ if $0 \leq p < p_c(d)$.

In particular, σ^ is discontinuous at $p = p_c(d)$.*

It is hard to say much about the value of $\sigma^*(p)$ at $p = p_c(d)$; see reference [33].

Proof (a) Consider percolation in the d -dimensional half-space $\mathbb{L}_+ \subset \mathbb{L}$ with vertices $\mathbb{Z}^{d-1} \times \mathbb{N}$ (where $\mathbb{N} = \{1, 2, 3, \dots\}$) and the corresponding edges of \mathbb{E} .

It is known that the critical density for homogeneous percolation in \mathbb{L}_+ is equal to $p_c(d)$ [4]. Thus, if C_1 is the connected component containing the vertex $(0, \dots, 0, 1)$ in the subgraph induced by the open edges of \mathbb{L}_+ , then $P_p^H(|C_1|=\infty) > 0$ for $p > p_c(d)$. Moreover, by also considering the status of the single edge from the origin to $(0, \dots, 0, 1)$, we have $P_{p,0}^I(|C(0)|=\infty) \geq p P_p^H(|C_1|=\infty)$. Hence $\theta^I(p, 0) > 0$ if $p > p_c(d)$. Part (a) follows.

(b) Fix $p < p_c(d)$. If $\sigma < p_c(d)$, then by monotonicity, $\theta^I(p, \sigma) \leq \theta^H(\max\{p, \sigma\}) = 0$ [since $\max\{p, \sigma\} < p_c(d)$]. This shows that $\sigma^*(p) \geq p_c(d)$. We obtain $\sigma^*(p) \leq \sigma^*(0) = p_c(s)$ from equation (13) and the fact that σ^* is non-increasing in p .

The phase boundary $\sigma^*(p)$ may be estimated for small p in a mean field approximation using the approach in reference [6]. Consider percolation in the defect lattice \mathbb{L}_0 , which has critical density $\sigma_c = p_c(s)$. An infinite cluster can grow in \mathbb{L}_0 either along edges $x \sim y \in \mathbb{L}_0$, or if such an edge is closed, then along a “bridge” of three edges in $\mathbb{L} \setminus \mathbb{L}_0$ in a \sqcap -shape, and joining x to y . That is, a bridge of $x \sim y$ is a sequence of three edges $x \sim r \sim t \sim y$ with $r, t \in \mathbb{L} \setminus \mathbb{L}_0$.

The probability that $x \sim y$ is open is σ , and the probability that a particular bridge of $x \sim y$ is open is p^3 . Since $x \sim y$ is bridged by $2(d-s)$ bridges, the probability that at least one of them is open is $1 - (1 - p^3)^{2(d-s)}$. Hence, the probability that either $x \sim y$ is open, or that it is closed and at least one of its bridges is open, is given by $\sigma + (1 - \sigma)(1 - (1 - p^3)^{2(d-s)})$.

An approximation is made by assuming that bridges of different edges in \mathbb{L}_0 are open or closed independently. In this approximation a cluster will grow to infinity along \mathbb{L}_0 using bridges if the density of open edges or closed edges with an open bridge is greater than σ_c , i.e. if

$$\sigma_c < \sigma + (1 - \sigma) \left(1 - (1 - p^3)^{2(d-s)}\right). \quad (19)$$

Solving for σ gives an estimate of $\sigma^*(p)$ for small p :

$$\sigma^*(p) \simeq \frac{\sigma_c + (1 - p^3)^{2(d-s)} - 1}{(1 - p^3)^{2(d-s)}} = \sigma_c - 2(d-s)(1 - \sigma_c)p^3 + O(p^6) \quad (20)$$

(the approximation of [6] is different only because they consider a half-space with \mathbb{L}_0 being the boundary plane). This result should be good for small values of p in particular, because the assumption that bridges are independent is better at low densities of edges in the bulk lattice.

2.1 Strict monotonicity of $\sigma^*(p)$

The next proposition serves two purposes. Firstly, it shows that σ^* is a strictly decreasing function for $p \in [0, p_c(d)]$. Secondly, it proves that the cubic form of the mean-field approximation of equation (20) (see reference [6]) is a rigorous upper bound for $\sigma^*(p)$ when p is close to 0.

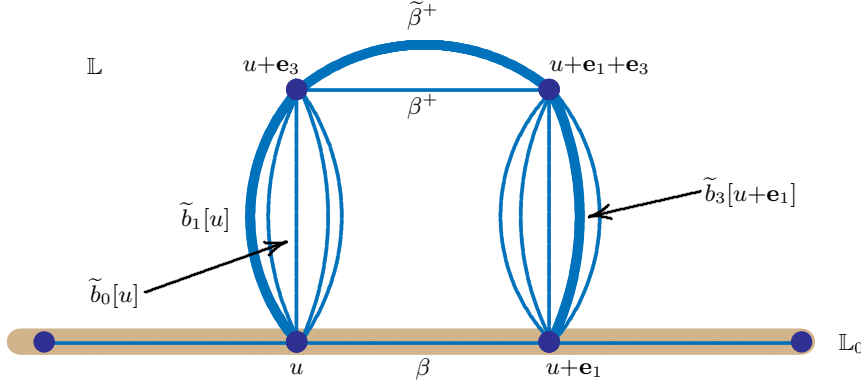


Fig. 4 \mathbb{L} is modified in the proof of proposition 2 by adding edges next to edges $\beta \in \mathbb{L}_0$ as shown above for the case $d = 3$, $s = 2$ and $J = 1$. The set $\mathcal{S}(\beta)$ consists of the three bold edges.

Proposition 2 Assume $2 \leq s < d$. Fix $0 \leq p < p_c(d)$. Then there is a positive constant A (possibly depending on p) such that

$$\sigma^*(p + \delta) \leq \sigma^*(p) - A\delta^3 \quad \text{for all sufficiently small positive } \delta.$$

In particular, σ^* is strictly decreasing on $[0, p_c(d)]$.

Proof Consider a modified lattice obtained by adding some new edges to \mathbb{L} as follows (see figure 4). For each vertex $u \in \mathbb{L}_0$, let $b_0[u]$ be the edge of \mathbb{E} joining u to $u + \mathbf{e}_d$. Introduce $2s$ new edges, $\tilde{b}_1[u], \dots, \tilde{b}_{2s}[u]$, each joining u to $u + \mathbf{e}_d$; thus we have $2s + 1$ parallel edges joining these two vertices. For each edge β in \mathbb{E}_0 , let $\beta^+ = \beta + \mathbf{e}_d$ (i.e., the edge in \mathbb{E} obtained by translating β one unit in the d^{th} coordinate direction). Also, let $\tilde{\beta}^+$ be a new edge parallel to β^+ (i.e., having the same endpoints). Let $\tilde{\mathbb{L}}$ be the (inhomogeneous) lattice with sites \mathbb{Z}^d and edges

$$\mathbb{E} \cup \{\tilde{b}_j[u] : u \in \mathbb{L}_0, 1 \leq j \leq 2s\} \cup \{\tilde{\beta}^+ : \beta \in \mathbb{E}_0\}.$$

Let $P_{p,\sigma,t}^*$ be the probability measure for bond percolation on $\tilde{\mathbb{L}}$ with parameters $p, \sigma, t \in [0, 1]$ so that edges are independent and

$$P_{p,\sigma,t}^*(e \text{ is open}) = \begin{cases} p & \text{if } e \in \mathbb{E} \setminus \mathbb{E}_0 \\ \sigma & \text{if } e \in \mathbb{E}_0 \\ t & \text{if } e = \tilde{\beta}^+ \text{ for } \beta \in \mathbb{E}_0 \\ 1 - (1 - t)^{1/2s} & \text{if } e = \tilde{\beta}_j[u], u \in \mathbb{L}_0, 1 \leq j \leq 2s. \end{cases}$$

With each edge $\beta \in \mathbb{E}_0$, we associate a set $\mathcal{S}(\beta)$ of three edges in $\tilde{\mathbb{L}}$ as follows. Let $u \in \mathbb{L}_0$ and $J \in \{1, \dots, s\}$ be such that β has endpoints u and $u + \mathbf{e}_J$. Then define

$$\mathcal{S}(\beta) = \{\tilde{\beta}^+, \tilde{b}_J[u], \tilde{b}_{J+s}[u + \mathbf{e}_J]\}.$$

Thus the edges of $\mathcal{S}(\beta)$ form a three-step path from one endpoint of β to the other. Note that for two distinct edges β_1 and β_2 in \mathbb{E}_0 , the sets $\mathcal{S}(\beta_1)$ and $\mathcal{S}(\beta_2)$ are disjoint. For each $\beta \in \mathbb{E}_0$, define the random variable $Y(\beta)$ to be 1 if either β is open, or if all three edges in $\mathcal{S}(\beta)$ are open; and define $Y(\beta)$ to be 0 otherwise. Then

$$\begin{aligned} P_{p,\sigma,t}^*(Y(\beta) = 1) &= \sigma + (1 - \sigma)t(1 - (1 - t)^{1/2s})^2 \\ &\geq \sigma + \frac{(1 - \sigma)t^3}{(2s)^2} \end{aligned}$$

where we used the fact that $tw \leq 1 - (1 - t)^w$ for $t, w \in (0, 1)$. Since $\mathcal{S}(\beta)$ is disjoint from $\mathbb{E} \setminus \mathbb{E}_0$, it follows that

$$P_{p,\sigma,t}^*(|C|=\infty) \geq \theta^I \left(p, \sigma + \frac{(1 - \sigma)t^3}{(2s)^2} \right). \quad (21)$$

Next, given a small positive δ , let $t = \delta/(1-p)$, so that $1 - (1-p)(1-t) = p + \delta$. Then for each $u \in \mathbb{L}_0$, we have

$$\begin{aligned} P_{p,\sigma,t}^*(\tilde{b}_j[u] \text{ is empty for all } j = 0, \dots, 2s) &= (1 - p) \left((1 - t)^{1/2s} \right)^{2s} \\ &= (1 - p)(1 - t) = 1 - (p + \delta). \end{aligned} \quad (22)$$

In addition,

$$P_{p,\sigma,t}^*(\beta^+ \text{ or } \tilde{\beta}^+ \text{ is occupied}) = 1 - (1 - p)(1 - t) = p + \delta. \quad (23)$$

Combining equations (22) and (23) shows that percolation on $\tilde{\mathbb{L}}$ governed by $P_{p,\sigma,t}^*$ is essentially the same as percolation on \mathbb{L} governed by a modification of $P_{p,\sigma}$ in which some edges of $\mathbb{E} \setminus \mathbb{E}_0$ have their density raised from p to $p + \delta$. This shows that

$$\theta^I(p + \delta, \sigma) \geq P_{p,\sigma,t}^*(|C|=\infty). \quad (24)$$

Combining equations (21) and (24) shows that $\theta^I(p + \delta, \sigma) > 0$ if $\sigma + (1 - \sigma)t^3/(2s)^2 > \sigma^*(p)$, which can be rearranged to yield the inequality

$$\sigma > \frac{\sigma^*(p) - t^3/(2s)^2}{1 - t^3/(2s)^2}. \quad (25)$$

Therefore the right-hand side of equation (25) is an upper bound for $\sigma^*(p + \delta)$. The proposition follows (using the fact that $\sigma^*(p) < 1$).

Corollary 1 $\sigma^*(p_c(d)) < p_c(s)$.

Proof By the strict monotonicity of σ^* (proposition 2) and equation (13), we have $\sigma^*(p_c(d)) < \sigma^*(0) = p_c(s)$.

3 Uniqueness of the critical point

It follows trivially from equation (10) that $\theta^I(p, \sigma) > 0$ implies that $\chi^I(p, \sigma) = \infty$. To show a converse is more difficult. We would like to prove that the percolation transition is at the same place as the transition from finite to infinite susceptibility. Such an assertion, often called the “uniqueness of the critical point”, is the subject of the following theorem, whose proof is the goal of this section.

Theorem 1 *Suppose that $\chi^I(p_1, \sigma_1) = \infty$. Then either*

- a) $\theta^I(p_1, \sigma_1) > 0$, or
- b) $\theta^I(p_1, \sigma_1) = 0$ and $\theta^I(p_1 + \Delta, \sigma_1 + \Delta) > 0$ for all small positive Δ ; in particular, (p_1, σ_1) is a boundary point of \mathcal{R}_0 .

By proposition 1(a), theorem 1 holds whenever $p_1 \geq p_c(d)$. If $p_1 \geq \sigma_1$, then $\chi^I(p_1, \sigma_1) = \infty$ implies that $\chi^H(p_1) = \infty$ (by monotonicity), which in turn implies that $p_1 \geq p_c(d)$ by our knowledge of the homogeneous case. Thus theorem 1 holds whenever $p_1 \geq \sigma_1$.

Consequently, for the rest of this section we shall assume that $p_1 < p_c(d)$, $p_1 < \sigma_1$, and $\chi^I(p_1, \sigma_1) = \infty$. If we also have $\chi^{f,I}(p_1, \sigma_1) < \infty$, then (by comparing equations (10) and (11)), we must have $\theta^I(p_1, \sigma_1) > 0$, and we are done. Therefore we shall also assume that $\chi^{f,I}(p_1, \sigma_1) = \infty$.

The proof for the homogeneous case in reference [1] (see also reference [17]) relies on augmenting the model to include a ghost vertex g . We follow a similar approach in the inhomogeneous case.

Thus we introduce the ghost vertex g and edges $\mathbb{E}_g = \{g \sim x \mid x \in \mathbb{L}\}$. Edges in \mathbb{E}_g are open with probability $\gamma \in (0, 1)$. Define \bar{G} to be the (random) collection of vertices in \mathbb{L} adjacent to g through open edges in \mathbb{E}_g .

Percolation on $\mathbb{E} \cup \mathbb{E}_g$ has parameters (p, σ, γ) . Since edges in \mathbb{E}_g are open with probability γ , it follows that

$$\theta^I(p, \sigma, \gamma) = 1 - \sum_{n=1}^{\infty} (1 - \gamma)^n P_{p, \sigma}^I(|C|=n) \quad (26)$$

is the probability that there is an open path from the origin to the ghost vertex g . Observe that by Abel’s theorem

$$\begin{aligned} \lim_{\gamma \rightarrow 0^+} \theta^I(p, \sigma, \gamma) &= 1 - \sum_{n=1}^{\infty} P_{p, \sigma}^I(|C|=n) \\ &= 1 - P_{p, \sigma}^I(|C| < \infty) = P_{p, \sigma}^I(|C| = \infty) = \theta^I(p, \sigma). \end{aligned}$$

Similarly, it is the case that

$$\chi^I(p, \sigma, \gamma) := \sum_{n=1}^{\infty} n (1 - \gamma)^n P_{p, \sigma}^I(|C|=n) = (1 - \gamma) \frac{\partial \theta^I}{\partial \gamma} \quad (27)$$

for $\gamma \in (0, 1)$. We also have

$$\lim_{\gamma \rightarrow 0^+} \chi^I(p, \sigma, \gamma) = \chi^{f,I}(p, \sigma). \quad (28)$$

If $p < \sigma$, then theorem 8 in the Appendix shows that the functions $\theta^I(p, \sigma, \gamma)$ and $\chi^I(p, \sigma, \gamma)$ satisfy the differential inequalities

$$\mathbf{q} \cdot \nabla \theta^I(p, \sigma, \gamma) \leq 2d \chi^H(p) \theta^I(p, \sigma, \gamma) (1 - \gamma) \frac{\partial \theta^I}{\partial \gamma}, \quad (29)$$

$$\begin{aligned} \theta^I(p, \sigma, \gamma) &\leq \gamma \frac{\partial \theta^I}{\partial \gamma} + (\theta^I(p, \sigma, \gamma))^2 \\ &\quad + \chi^H(p) \theta^I(p, \sigma, \gamma) (\mathbf{p} \cdot \nabla \theta^I(p, \sigma, \gamma)) \end{aligned} \quad (30)$$

where $\nabla = (\frac{\partial}{\partial p}, \frac{\partial}{\partial \sigma})$, $\mathbf{p} = (p, \sigma)$ and $\mathbf{q} = (1 - p, 1 - \sigma)$.

Theorem 2 Assume that $p < p_c(d)$, $p \leq \sigma$, and $\chi^{f,I}(p, \sigma) = \infty$. Then there exists an $\alpha = \alpha(p, \sigma)$ such that

$$\theta^I(p, \sigma, \gamma) \geq \alpha \gamma^{1/2}$$

for all small positive values of γ .

Proof Suppose that $p < p_c(d)$, $0 < p \leq \sigma < 1$, and $\chi^{f,I}(p, \sigma) = \infty$. If $\theta^I(p, \sigma) > 0$ then the theorem is trivial. Thus, assume that $\theta^I(p, \sigma) = 0$. This implies that $\lim_{\gamma \rightarrow 0^+} \theta^I(p, \sigma, \gamma) = 0$.

Observe that since $p \leq \sigma$ and $1 - p \geq 1 - \sigma$,

$$\begin{aligned} \mathbf{p} \cdot \nabla \theta^I &= p \frac{\partial \theta^I}{\partial p} + \sigma \frac{\partial \theta^I}{\partial \sigma} \leq \frac{\sigma}{1 - \sigma} \left((1 - p) \frac{\partial \theta^I}{\partial p} + (1 - \sigma) \frac{\partial \theta^I}{\partial \sigma} \right) \\ &= \frac{\sigma}{1 - \sigma} \mathbf{q} \cdot \nabla \theta^I. \end{aligned} \quad (31)$$

With p and σ fixed, put $\theta^I(p, \sigma, \gamma) = f(\gamma)$. The properties of $f(\gamma)$ are such that f is strictly increasing and continuously differentiable on $(0, 1)$ with $f(0) = 0$ and $f(1) = 1$. Using equation (31) to eliminate $\nabla \theta^I$ from the differential inequalities (29) and (30), we obtain

$$f(\gamma) \leq \gamma \frac{\partial f(\gamma)}{\partial \gamma} + f^2(\gamma) + \frac{2d\sigma}{1 - \sigma} [\chi^H(p) f(\gamma)]^2 (1 - \gamma) \frac{\partial f(\gamma)}{\partial \gamma}. \quad (32)$$

By the mean value theorem there exists a $\psi \in (0, \gamma)$ such that

$$f'(\psi) = \frac{1}{\gamma} f(\gamma).$$

As $\gamma \rightarrow 0^+$, $\psi \rightarrow 0^+$, so that by equations (27) and (28),

$$\lim_{\gamma \rightarrow 0^+} \frac{\gamma}{f(\gamma)} = 0.$$

Define the inverse function of f to be h . Then h is strictly increasing and continuously differentiable with $h(0) = 0$ and $h(1) = 1$, and satisfying

$$\lim_{\phi \rightarrow 0^+} \frac{h(\phi)}{\phi} = 0. \quad (33)$$

This shows that $h'(\phi)$ is bounded on $(0, \Phi]$ for some $\Phi > 0$. Also, note that

$$\frac{dh}{d\phi} = \left(\frac{df}{d\gamma} \right)^{-1}.$$

By substituting $\gamma = h(\phi)$ and $f(\gamma) = \phi$ in equation (32) and simplifying, we get

$$\frac{1}{\phi} \frac{dh}{d\phi} - \frac{h}{\phi^2} \leq \frac{2d\sigma}{1-\sigma} (\chi^H(p))^2 (1-h) + \frac{dh}{d\phi}. \quad (34)$$

Observe that h is a strictly increasing function with bounds $0 \leq h(\phi) \leq 1$ where $h(0) = 0$ and $h(1) = 1$. Furthermore, since $h'(\phi)$ is bounded on $(0, \Phi]$, equation (34) implies that there exists a $\beta(p, \sigma) > 0$ such that

$$\frac{1}{\phi} \frac{dh}{d\phi} - \frac{h}{\phi^2} \leq \beta, \quad \text{if } 0 < \phi \leq \Phi.$$

Integrate this over $\phi \in (0, x)$ where $x \leq \Phi$ to get

$$\frac{h(\phi)}{\phi} \Big|_0^x \leq \beta x.$$

By equation (33), this gives $h(x) \leq \beta x^2$ for all $x \in (0, \Phi]$. In terms of f this says $\gamma \leq \beta (f(\gamma))^2$ for $\gamma \leq h(\Phi)$, or $f(\gamma) \geq \alpha \gamma^{1/2}$ where $\alpha = \beta^{-1/2}$.

We shall now complete the proof of theorem 1 in the remaining situation of interest, namely that $p_1 < p_c(d)$, $p_1 < \sigma_1$ and $\chi^{f,I}(p_1, \sigma_1) = \infty$. Let $\theta^I \equiv \theta^I(p, \sigma, \gamma)$ and $\kappa = \kappa(p) = \chi^H(p)$. Write equation (30) as

$$0 \leq \frac{1}{\theta^I} \frac{\partial \theta^I}{\partial \gamma} + \frac{1}{\gamma} (\theta^I - 1 + \kappa \mathbf{p} \cdot \nabla \theta^I). \quad (35)$$

If $\theta^I(p_1, \sigma_1) > 0$ then the proof is done, so assume that $\theta^I(p_1, \sigma_1) = 0$. Let $\Delta > 0$ be small, and define $\bar{p} = p_1 + \Delta/2$ and $\bar{\sigma} = \sigma_1 + \Delta/2$. Put $p = p(t) = p_1 + t/2$ and $\sigma = \sigma(t) = \sigma_1 + t/2$, and define $\mathbf{p}(t) = (p(t), \sigma(t))$.

We shall follow the method as presented in reference [17] to show that $\theta^I(\bar{p}, \bar{\sigma}) > 0$. We begin with the following claim. (Notice that $\kappa = \chi^H(p(t))$ and $\theta^I = \theta^I(p(t), \sigma(t))$ now depend on t .)

$$\text{Claim: } \theta^I - 1 + \kappa \mathbf{p}(t) \cdot \nabla \theta^I \leq \frac{d}{dt} ((p(t) + \sigma(t))(2\kappa \theta^I - 1)).$$

Proof of claim:

$$\begin{aligned} \frac{d}{dt} ((p(t) + \sigma(t))(2\kappa \theta^I - 1)) &= 2\kappa \theta^I - 1 + (p(t) + \sigma(t)) \frac{d}{dt} (2\kappa \theta^I) \\ &= 2\kappa \theta^I - 1 + (p(t) + \sigma(t)) \left(\kappa' \theta^I + \kappa \left(\frac{\partial}{\partial p} \theta^I + \frac{\partial}{\partial \sigma} \theta^I \right) \right) \\ &\geq \theta^I - 1 + \kappa(p(t) + \sigma(t)) \left(\frac{\partial}{\partial p} \theta^I + \frac{\partial}{\partial \sigma} \theta^I \right) \\ &\geq \theta^I - 1 + \kappa \mathbf{p}(t) \cdot \nabla \theta^I \end{aligned}$$

since $\kappa \geq 1$, $\kappa' \geq 0$ and $\theta^I \in [0, 1]$ and is non-decreasing with p and σ . This completes the proof of the claim.

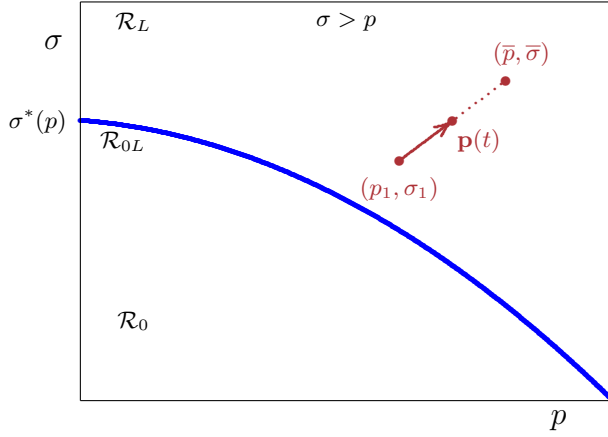


Fig. 5 Integrating equation (36) for $t \in (0, \Delta)$ takes (p_1, σ_1) to $(\bar{p}, \bar{\sigma})$.

By the claim, equation (35) implies

$$0 \leq \frac{1}{\theta^I} \frac{\partial \theta^I}{\partial \gamma} + \frac{1}{\gamma} \frac{d}{dt} ((p(t) + \sigma(t))(2\kappa \theta^I - 1)). \quad (36)$$

Integrate the first term with respect to $\gamma \in (\delta, \epsilon)$ to obtain

$$\int_{\delta}^{\epsilon} \frac{1}{\theta^I} \frac{\partial \theta^I}{\partial \gamma} d\gamma = \log \theta^I(p(t), \sigma(t), \epsilon) - \log \theta^I(p(t), \sigma(t), \delta).$$

Now integrate the result with respect to $t \in (0, \Delta)$, and use the bounds

$$\theta^I(p_1, \sigma_1, \gamma) \leq \theta^I(p(t), \sigma(t), \gamma) \leq \theta^I(\bar{p}, \bar{\sigma}, \gamma)$$

on θ^I to get

$$\int_0^{\Delta} \left[\int_{\delta}^{\epsilon} \frac{1}{\theta^I} \frac{\partial \theta^I}{\partial \gamma} d\gamma \right] dt \leq \Delta (\log \theta^I(\bar{p}, \bar{\sigma}, \epsilon) - \log \theta^I(p_1, \sigma_1, \delta)). \quad (37)$$

We integrate and bound the second term in equation (36) similarly, as follows. The integral with respect to t is straightforward, and integrating the result with respect to γ gives

$$\begin{aligned} \int_{\delta}^{\epsilon} [(\bar{p} + \bar{\sigma}) (2\kappa(\bar{p})\theta^I(\bar{p}, \bar{\sigma}, \gamma) - 1) - (p_1 + \sigma_1) (2\kappa(p_1)\theta^I(p_1, \sigma_1, \gamma) - 1)] \frac{d\gamma}{\gamma} \\ \leq \int_{\delta}^{\epsilon} [(\bar{p} + \bar{\sigma}) (2\kappa(\bar{p})\theta^I(\bar{p}, \bar{\sigma}, \epsilon) - 1) + (p_1 + \sigma_1)] \frac{d\gamma}{\gamma} \\ = \log(\epsilon/\delta) (2(\bar{p} + \bar{\sigma}) \kappa(\bar{p})\theta^I(\bar{p}, \bar{\sigma}, \epsilon) - \Delta) \end{aligned}$$

since $(\bar{p} + \bar{\sigma}) - (p_1 + \sigma_1) = \Delta$. Using this and equation (37), the inequality in (36) becomes

$$0 \leq \frac{\Delta (\log \theta^I(\bar{p}, \bar{\sigma}, \epsilon) - \log \theta^I(p_1, \sigma_1, \delta))}{\log \epsilon - \log \delta} + 2(\bar{p} + \bar{\sigma}) \kappa(\bar{p}) \theta^I(\bar{p}, \bar{\sigma}, \epsilon) - \Delta.$$

By theorem 2, $\theta^I(p_1, \sigma_1, \gamma) \geq \alpha \gamma^{1/2}$. Hence $-\log \theta^I(p_1, \sigma_1, \delta) \leq -\log \alpha - \frac{1}{2} \log \delta$ for small δ . Substituting this into the above and taking $\delta \rightarrow 0^+$ gives

$$0 \leq \frac{\Delta}{2} + 2(\bar{p} + \bar{\sigma}) \kappa(\bar{p}) \theta^I(\bar{p}, \bar{\sigma}, \epsilon) - \Delta$$

which can be rearranged to give

$$\frac{\Delta}{2} \leq 4\kappa(\bar{p}) \theta^I(\bar{p}, \bar{\sigma}, \epsilon).$$

Taking $\epsilon \rightarrow 0^+$ completes the proof.

4 Exponential decay of the cluster size distribution

In this section the exponential decay of $P_{p,\sigma}^I(|C| = n)$ in the subcritical phase is examined (corresponding to region \mathcal{R}_0 in figure 3). Proving this follows the same general outline as for the similar result in homogeneous percolation, with minor modifications.

The two-point connectivity function is defined by

$$\tau_{p,\sigma}^I(x, y) = E_{p,\sigma}^I \mathbb{1}_{\{x \leftrightarrow y\}} \quad (38)$$

where $\mathbb{1}_{\{x \leftrightarrow y\}}$ is the indicator function of the event that there exists an open path joining vertices x and y in \mathbb{Z}^d (or the indicator function of the event that x and y belong to the same open cluster).

Naturally, the number of vertices in the cluster at the origin is $|C| = \sum_x \mathbb{1}_{\{0 \leftrightarrow x\}}$ and so the susceptibility defined in equation (10) may be expressed in terms of the two point connectivity function via

$$\chi^I(p, \sigma) = E_{p,\sigma}^I |C| = E_{p,\sigma}^I \sum_x \mathbb{1}_{\{0 \leftrightarrow x\}} = \sum_x E_{p,\sigma}^I \mathbb{1}_{\{0 \leftrightarrow x\}} = \sum_x \tau_{p,\sigma}^I(0, x).$$

More generally, consider the open cluster $C(y)$ at the site y and define

$$\begin{aligned} \chi^I(p, \sigma; y) &= E_{p,\sigma}^I |C(y)| = E_{p,\sigma}^I \sum_x \mathbb{1}_{\{y \leftrightarrow x\}} \\ &= \sum_x E_{p,\sigma}^I \mathbb{1}_{\{y \leftrightarrow x\}} = \sum_x \tau_{p,\sigma}^I(y, x). \end{aligned}$$

By lemma 6 and equation (94) in the appendix,

$$\sum_x \tau_{p,\sigma}^I(y, x) = \chi^I(p, \sigma; y) \leq \chi^H(p) \chi^I(p, \sigma; 0) \quad \text{for every } y \in \mathbb{Z}^d. \quad (39)$$

A similar and generalised bound on the $(n+1)$ -point connectivity function $\tau_{p,\sigma}^I(y, x_1, x_2, x_3, \dots, x_n)$ defined by

$$\begin{aligned} &\tau_{p,\sigma}^I(y, x_1, x_2, x_3, \dots, x_n) \\ &= E_{p,\sigma}^I (\mathbb{1}_{\{y \leftrightarrow x_1\}} \mathbb{1}_{\{y \leftrightarrow x_2\}} \mathbb{1}_{\{y \leftrightarrow x_3\}} \dots \mathbb{1}_{\{y \leftrightarrow x_n\}}) \end{aligned} \quad (40)$$

should be determined. This will give an upper bound on $E_{p,\sigma}^I |C|^n$ since

$$E_{p,\sigma}^I |C|^n = \sum_{x_1, x_2, \dots, x_n} \tau_{p,\sigma}^I(0, x_1, x_2, x_3, \dots, x_n). \quad (41)$$

In the case of the three-point connectivity function, the arguments given in Chapter 6 of reference [17] can be extended to apply to the inhomogeneous model considered in this paper. As such, we can state the following lemma without proof.

Lemma 1 *For all values of p and σ and vertices x_0, x_1 and x_2 ,*

$$\tau_{p,\sigma}^I(x_0, x_1, x_2) \leq \sum_y \tau_{p,\sigma}^I(y, x_0) \tau_{p,\sigma}^I(y, x_1) \tau_{p,\sigma}^I(y, x_2).$$

Thus, by equation (39),

$$E_{p,\sigma}^I |C|^2 \leq \sum_{y, x_1, x_2} \tau_{p,\sigma}^I(y, 0) \tau_{p,\sigma}^I(y, x_1) \tau_{p,\sigma}^I(y, x_2) \leq (\chi^H(p) \chi^I(p, \sigma; 0))^3.$$

The generalisation of the above bound for the inhomogeneous model proceeds along the same line as the argument given by Aizenman and Newman [3] for the case of homogeneous percolation, involving the characterisation of connectivity functions as sums over labeled skeletons (trees with all interior vertices of degree three) [17].

Following the arguments for the homogeneous case one arrives at the bound

$$\tau_{p,\sigma}^I(x_0, x_1, \dots, x_n) \leq \sum_S \sum_{\psi_x} \prod_{u \sim v \in S} \tau_{p,\sigma}^I(\psi_x(u), \psi_x(v)) \quad (42)$$

in the notation of reference [17]. The summation over S is over all labeled skeletons with $n+1$ exterior vertices (or *end vertices*). The summation over ψ_x is a sum over all admissible mappings from the vertex set of a skeleton S into \mathbb{Z}^d (this is a summation over all possible $\psi_x(v)$ as v takes on values in the interior vertices of S). The product is over all branches $u \sim v$ (edges joining adjacent vertices u and v in the graph theoretic sense) of S .

Equation (42) must be summed over x_j for $1 \leq j \leq n$ to obtain an upper bound on $E_{p,\sigma}^I |C|^n$. Since the x_i are end-vertices in S , and are vertices in the two-point functions, one may use equation (39) to bound these summations from above. That reduces equation (42) to

$$E_{p,\sigma}^I |C|^n \leq (\chi^H(p) \chi^I(p, \sigma))^n \sum_S \sum_{\psi} \prod'_{u \sim v} \tau_{p,\sigma}^I(\psi(u), \psi(v))$$

and the primed product is only over branches $u \sim v$ where u and v are vertices in the skeleton which are either the origin, or are interior vertices of S . Performing the summation over ψ and using equation (39) as a bound gives

$$E_{p,\sigma}^I |C|^n \leq N_{n+1} (\chi^H(p) \chi^I(p, \sigma))^{2n-1} \quad (43)$$

where

$$N_{n+1} = \frac{(2n-2)!}{2^{n-1}(n-1)!} \quad (44)$$

is the number of labeled skeletons with $n+1$ exterior (or end-)vertices. This is the generalisation of lemma 1.

The bound in equation (43) is sufficient for the following theorem.

Theorem 3 *Suppose that $\chi^I(p, \sigma) < \infty$. Then for every n ,*

$$P_{p,\sigma}(|C| \geq n) \leq 2e^{-n/(2(\chi^H(p)\chi^I(p,\sigma))^2)}. \quad (45)$$

Proof The proof follows the approach in Grimmett [17]. Use equation (43) to see that

$$\begin{aligned} E_{p,\sigma}^I(|C|e^{t|C|}) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} E_{p,\sigma}^I(|C|^{n+1}) \\ &\leq (\chi^H(p)\chi^I(p,\sigma)) \left[1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} N_{n+2} (\chi^H(p)\chi^I(p,\sigma))^{2n} \right] \\ &= \frac{\chi^H(p)\chi^I(p,\sigma)}{\sqrt{1 - 2t(\chi^H(p)\chi^I(p,\sigma))^2}} \end{aligned}$$

whenever $t \in [0, \frac{1}{2}(\chi^H(p)\chi^I(p,\sigma))^{-2}]$. Markov's inequality [19] then shows that

$$P_{p,\sigma}^I(|C| \geq n) = P_{p,\sigma}^I(|C|e^{t|C|} \geq ne^{tn}) \leq \frac{1}{ne^{tn}} E_{p,\sigma}^I(|C|e^{t|C|}).$$

This shows that

$$P_{p,\sigma}^I(|C| \geq n) \leq \frac{\chi^H(p)\chi^I(p,\sigma)}{ne^{tn}\sqrt{1 - 2t(\chi^H(p)\chi^I(p,\sigma))^2}}.$$

The final step in the proof is to choose an appropriate value for t . The last inequality is valid for $0 \leq t < \frac{1}{2}(\chi^H(p)\chi^I(p,\sigma))^{-2}$, so put

$$t = \frac{1}{2(\chi^H(p)\chi^I(p,\sigma))^2} - \frac{1}{2n}.$$

If $n > (\chi^H(p)\chi^I(p,\sigma))^2$ then $t > 0$, and with this choice of t one gets

$$P_{p,\sigma}^I(|C| \geq n) \leq \frac{\sqrt{e}}{\sqrt{n}} e^{-\frac{n}{2(\chi^H(p)\chi^I(p,\sigma))^2}}$$

and equation (45) follows. Finally, (45) is trivially true for $n \leq (\chi^H(p)\chi^I(p,\sigma))^2$ because $2e^{-1/2} > 1$. This completes the proof.

Since for each cluster C one has $\frac{1}{d}\|C\| \leq |C| \leq \|C\| + 1$, it follows that $P_{p,\sigma}^I(\|C\| \geq n) \leq P_{p,\sigma}^I(|C| \geq n/d)$ and we get the following a corollary of theorem 3:

Corollary 2 *Suppose that (p, σ) is in the interior of \mathcal{R}_0 . Then there exists a function $\lambda^I(p, \sigma) > 0$ such that*

$$P_{p,\sigma}^I(\|C\|=n) \leq P_{p,\sigma}^I(\|C\|\geq n) \leq e^{-n\lambda^I(p,\sigma)}.$$

5 The supercritical region

5.1 Subexponential decay of the supercritical cluster size distribution

It is a result of homogeneous percolation that $P_p^H(|C|=n)$ does not decay exponentially with n in the supercritical phase. Instead, it is known that there exists a $\gamma_H(p) > 0$ such that

$$P_p^H(|C|=n) \geq e^{-\gamma_H(p)n^{(d-1)/d}} \quad \text{if } p > p_c(d). \quad (46)$$

A similar result [equation (15)] can be shown for the inhomogeneous model with $p > p_c(d)$ by considering percolation in the half-space \mathbb{L}^+ [31]. We state this as the following theorem and defer its proof to section 6.2.

Theorem 4 *If $p > p_c(d)$, then there exists a $\gamma_I(p) > 0$ such that*

$$P_{p,\sigma}^I(\infty > |C| \geq n) \geq P_{p,\sigma}^I(|C|=n) \geq e^{-\gamma_I(p)n^{(d-1)/d}}.$$

In the case that $p < p_c(d)$ and $\theta^I(p, \sigma) > 0$, the decay of the cluster size distribution has a different subexponential lower bound, which we state in theorem 5. We prove it using a variation of the method for homogeneous percolation due to Aizenman, Delyon and Souillard [2].

Theorem 5 *Assume that $0 < p < p_c(d)$ and that $\theta^I(p, \sigma) > 0$. Then there exist positive constants β_1 and β_2 (which are functions of (p, σ)) such that for all sufficiently large n ,*

$$P_{p,\sigma}^I(\infty > |C| \geq n) \geq \beta_1 e^{-\beta_2 n^{(s-1)/s} (\log^2 n)^{d-s}}. \quad (47)$$

To prove this result, we shall show that if $p < p_c(d)$ and $\theta^I(p, \sigma) > 0$, then there exist positive constants α_1 , α_2 and α_3 (depending on p and σ) such that

$$P_{p,\sigma}^I(\infty > |C| \geq \alpha_1 m^s) \geq \alpha_2 e^{-\alpha_3 m^{s-1} h(m)^{d-s}} \quad \text{for all sufficiently large } m, \quad (48)$$

where $h(m) = \lceil \log^2 m \rceil$. Theorem 5 follows from this by putting $m = (n/\alpha_1)^{1/s}$. The rest of this subsection is devoted to proving equation (48), with $h(m)$ being any function that grows faster than $\log m$ and slower than m .

Assume that $p < p_c(d)$ and $\theta^I(p, \sigma) > 0$.

Let $h : \mathbb{N} \rightarrow \mathbb{N}$ be a specified function satisfying $h(m) = o(m)$ and $\log m = o(h(m))$. For each $m \in \mathbb{N}$, define the rectangular box $B^*(m)$ centered at the origin in \mathbb{L} by

$$\begin{aligned} B^*(m) &= ([-m, m]^s \times [-h(m), h(m)]^{d-s}) \cap \mathbb{L} \\ &= \left\{ z \in \mathbb{Z}^d \mid |z_i| \leq m \text{ for } i = 1, \dots, s \text{ and } |z_i| \leq h(m) \text{ for } i = s+1, \dots, d \right\}. \end{aligned} \quad (49)$$

We separate the boundary of $B^*(m)$ into a vertical part $\partial_{\text{vert}}(m)$ and a horizontal part $\partial_{\text{hor}}(m)$:

$$\begin{aligned} \partial_{\text{vert}}(m) &= \left\{ v \in B^*(m) \mid |v_i| = m \text{ for at least one } i \leq s \right\}, \quad \text{and} \\ \partial_{\text{hor}}(m) &= \left\{ v \in B^*(m) \setminus \partial_{\text{vert}}(m) \mid |v_i| = h(m) \text{ for at least one } i > s \right\}. \end{aligned} \quad (50)$$

Also, let $\partial_e(m)$ be the set of edges outside $B^*(m)$ incident on $\partial_{\text{vert}}(m)$:

$$\partial_e(m) = \{x \sim y \in \mathbb{E} : x \in \partial_{\text{vert}}(m), y \notin B^*(m)\}.$$

Given $v \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$, denote the event that v is connected to a point of A by an open path by $\{v \leftrightarrow A\} = \cup_{x \in A} \{v \leftrightarrow x\}$. For every $v \notin \mathbb{L}_0$,

$$P_{p,\sigma}^I \{v \leftrightarrow \mathbb{L}_0\} = P_p^H \{v \leftrightarrow \mathbb{L}_0\}. \quad (51)$$

Notice that if $p < p_c(d)$, then there exists a positive constant δ_p such that

$$P_p^H \{v \leftrightarrow \mathbb{L}_0\} \leq e^{-\delta_p \text{dist}(v, \mathbb{L}_0)} \quad (52)$$

where $\text{dist}(v, \mathbb{L}_0) = \min\{\|v - x\| \mid x \in \mathbb{L}_0\}$ (and $\|\cdot\|$ is the Euclidean norm). This is a consequence of the exponential decay of the cluster size distribution in homogeneous percolation if $p < p_c$ (see for example reference [3]).

Since $\text{dist}(v, \mathbb{L}_0) \geq h(m)$ if $v \in \partial_{\text{hor}}(m)$, the following lemma for the probability of the event $\{v \leftrightarrow \mathbb{L}_0\}$ follows from equations (51) and (52).

Lemma 2 *If $p < p_c(d)$, then there is a $\delta_p > 0$ such that*

$$P_{p,\sigma}^I \{v \leftrightarrow \mathbb{L}_0\} \leq e^{-\delta_p h(m)} \quad \text{for every } v \in \partial_{\text{hor}}(m).$$

For $x \in \mathbb{L}_0 \cap B^*(m)$, let $Q_m(x)$ be the event that there is an open path in $B^*(m)$ from x to $\partial_{\text{vert}}(m)$. Let F_m be the event that there is no open path from $\partial_{\text{hor}}(m)$ to \mathbb{L}_0 whose edges are all outside of $B^*(m) \cup \partial_e(m)$.

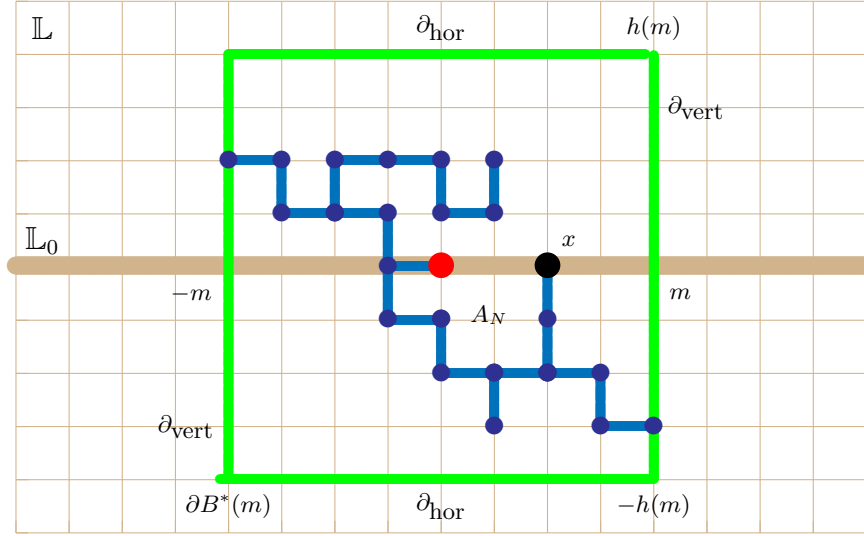


Fig. 6 The event $Q_m(x)$ that there is an open path in $B^*(m)$ from $x \in \mathbb{L}_0$ to $\partial_{\text{vert}}(m)$.

Lemma 3 Assume $p < p_c(d)$ and $\theta^I(p, \sigma) > 0$. Then for sufficiently large m ,

$$P_{p,\sigma}^I(Q_m(x)) > \frac{1}{2}\theta^I(p, \sigma) \quad \text{for all } x \in \mathbb{L}_0 \cap B^*(m). \quad (53)$$

Also,

$$\lim_{m \rightarrow \infty} P_{p,\sigma}^I(F_m) = 1. \quad (54)$$

Proof Let $Z_m = \{v \leftrightarrow \mathbb{L}_0 \text{ for some } v \in \partial_{\text{hor}}(m)\}$. Using $|\partial_{\text{hor}}(m)| = o(m^d)$ and $\log m = o(h(m))$, we observe that as $m \rightarrow \infty$,

$$P_{p,\sigma}^I(Z_m) \leq \sum_{v \in \partial_{\text{hor}}(m)} P_{p,\sigma}^I(v \leftrightarrow \mathbb{L}_0) \leq |\partial_{\text{hor}}(m)| e^{-\delta_p h(m)} = o(1). \quad (55)$$

Equation (54) follows since $F_m^c \subseteq Z_m$. For every $x \in \mathbb{L}_0 \cap B^*(m)$ we have $\{C(x) \cap \partial_{\text{hor}}(m) \neq \emptyset\} \subseteq Z_m$, so equation (55) implies that

$$\lim_{m \rightarrow \infty} \left(\max_{x \in \mathbb{L}_0 \cap B^*(m)} P_{p,\sigma}^I(C(x) \cap \partial_{\text{hor}}(m) \neq \emptyset) \right) = 0. \quad (56)$$

Next, for every $x \in \mathbb{L}_0 \cap B^*(m)$ we clearly have

$$\{|C(x)| = \infty\} \cap \{C(x) \cap \partial_{\text{hor}}(m) = \emptyset\} \subseteq Q_m(x).$$

Hence by equation (56), for sufficiently large m we have

$$P_{p,\sigma}^I(Q_m(x)) > \frac{1}{2}P_{p,\sigma}^I(|C(x)| = \infty) = \frac{1}{2}\theta^I(p, \sigma) \quad \text{for all } x \in \mathbb{L}_0 \cap B^*(m),$$

which proves Equation (53).

Let $L(m)$ be the set of vertices in $B^*(m)$ where $Q_m(x)$ occurs:

$$L(m) = \{x \in \mathbb{L}_0 \cap B^*(m) \mid Q_m(x) \text{ occurs}\}. \quad (57)$$

Then the next lemma shows that $P_{p,\sigma}^I(|L(m)| \geq \frac{1}{2}m^s\theta^I(p,\sigma))$ is not small in the supercritical regime.

Lemma 4 *Assume $p < p_c(d)$ and $\theta^I(p,\sigma) > 0$. Then for large m we have $P_{p,\sigma}^I(|L(m)| \geq \frac{1}{2}m^s\theta^I(p,\sigma)) \geq \frac{1}{4}\theta^I(p,\sigma)$.*

Proof Notice that $|L(m)| \leq |\mathbb{L}_0 \cap B^*(m)| = (2m+1)^s$. Moreover,

$$E_{p,\sigma}^I|L(m)| = \sum_{x \in \mathbb{L}_0 \cap B^*(m)} P_{p,\sigma}^I(Q_m(x)) \geq \frac{1}{2}\theta^I(p,\sigma)(2m+1)^s$$

for large m , by lemma 3. Hence,

$$\begin{aligned} \frac{1}{2}\theta^I(p,\sigma)(2m+1)^s &\leq E_{p,\sigma}^I|L(m)| \\ &\leq \frac{1}{2}m^s\theta^I(p,\sigma)P_{p,\sigma}^I(|L(m)| < \frac{1}{2}m^s\theta^I(p,\sigma)) \\ &\quad + (2m+1)^sP_{p,\sigma}^I(|L(m)| \geq \frac{1}{2}m^s\theta^I(p,\sigma)) \\ &\leq \frac{1}{2}m^s\theta^I(p,\sigma) + (2m+1)^sP_{p,\sigma}^I(|L(m)| \geq \frac{1}{2}m^s\theta^I(p,\sigma)). \end{aligned}$$

Solving for $P_{p,\sigma}^I(|L(m)| \geq \frac{1}{2}m^s\theta^I(p,\sigma))$ then gives

$$P_{p,\sigma}^I(|L(m)| \geq \frac{1}{2}m^s\theta^I(p,\sigma)) \geq \frac{1}{2}\theta^I(p,\sigma)\left(1 - \left(\frac{m}{2m+1}\right)^s\right) \geq \frac{1}{4}\theta^I(p,\sigma).$$

This completes the proof of the lemma.

Let A_m be the event that all edges in $\partial_{\text{vert}}(m)$ are open, and all edges of $\partial_e(m)$ are closed. Then, for some $\beta_2 > 0$,

$$P_{p,\sigma}^I(A_m) = e^{-O(|\partial_{\text{vert}}(m)|)} \geq e^{-\beta_2 m^{s-1}(h(m))^{d-s}}. \quad (58)$$

Let D_m be the event that the number of vertices in $L(m)$ is at least $\frac{1}{2}m^s\theta^I(p,\sigma)$, i.e.

$$D_m = \{|L(m)| \geq \frac{1}{2}m^s\theta^I(p,\sigma)\} \quad (59)$$

(recall that $L(m)$ is the set of vertices in $\mathbb{L}_0 \cap B^*(m)$ that are connected by open paths in $B^*(m)$ to $\partial_{\text{vert}}(m)$). Observe the following:

(i)

$$A_m \cap Q_m(0) \cap D_m \subseteq \{|C(0)| \geq \frac{1}{2}m^s\theta^I(p,\sigma)\}.$$

In other words, if all the edges in $\partial_{\text{vert}}(m)$ are open and all edges of $\partial_e(m)$ are closed, and if $Q_m(0)$ and D_m occur, then the cluster at the origin has size at least $\frac{1}{2}m^s\theta^I(p,\sigma)$.

(ii)

$$A_m \cap F_m \subseteq \{C(0) \cap \mathbb{L}_0 \subseteq B^*(m)\} \subseteq \{|C(0)| < \infty\} \cup Y_m, \text{ where}$$

$$Y_m = \{|C(0)| = \infty \text{ and } C(0) \cap \mathbb{L}_0 \subseteq B^*(m)\}.$$

- (iii) The events A_m , F_m , and $D_m \cap Q_m(0)$ are independent.
 (iv) By the FKG Inequality, and Lemmas 3 and 4, for large m we have

$$P_{p,\sigma}^I(Q_m(0) \cap D_m) \geq P_{p,\sigma}^I(Q_m(0)) P_{p,\sigma}^I(D_m) \geq \frac{1}{8} (\theta^I(p, \sigma))^2.$$

- (v) $P_{p,\sigma}^I(Y_m) = 0$, where Y_m was defined in (ii). To see this, consider the new percolation measure P^* in which each bond of $\mathbb{E}_0 \cap B^*(m+1)$ is open with probability σ , and every other bond is open with probability p . Then $P_{p,\sigma}^I(Y_m) = P^*(Y_m)$. Moreover, since $p < p_c(d)$, we have $P_p^H(Y_m) = 0$, and hence $P^*(Y_m)$ is also 0 because P_p^H and P^* differ on only finitely many edges.

Thus we conclude

$$\begin{aligned} P_{p,\sigma}^I & (\infty > |C(0)| \geq \tfrac{1}{2} m^s \theta^I(p, \sigma)) \\ & \geq P_{p,\sigma}^I(A_m \cap Q_m(0) \cap D_m \cap F_m \cap Y_m^c) \quad (\text{by (i) and (ii)}) \\ & = P_{p,\sigma}^I(A_m \cap Q_m(0) \cap D_m \cap F_m) \quad (\text{by (v)}) \\ & = P_{p,\sigma}^I(A_m) P_{p,\sigma}^I(D_m \cap Q_m(0)) P_{p,\sigma}^I(F_m) \quad (\text{by (iii)}). \end{aligned}$$

The proof of theorem 5 is completed by comparing this last lower bound with equations (58) and (54), as well as (iv).

5.2 Long-range connectivity above $p_c(d)$

Recall that the two-point connectivity function is $\tau_{p,\sigma}^I(x, y) = P_{p,\sigma}^I(x \leftrightarrow y)$. The next result shows that this function is bounded away from 0 for any given (p, σ) in \mathcal{R}_H .

Proposition 3 *Fix $p > p_c(d)$ and $\sigma \in [0, 1]$. Then $\inf\{\tau_{p,\sigma}^I(x, y) : x, y \in \mathbb{L}\} > 0$.*

Proof Let $\mathbf{0}$ be the origin in \mathbb{L} . Since $P_{p,\sigma}^I(x \leftrightarrow y) \geq P_{p,\sigma}^I(x \leftrightarrow \mathbf{0}) P_{p,\sigma}^I(x \leftrightarrow \mathbf{0})$ by the FKG inequality, it suffices to prove that $\inf\{P_{p,\sigma}^I(v \leftrightarrow \mathbf{0}) : v \in \mathbb{L}\} > 0$.

For $\mathbf{v} \in \mathbb{Z}^d$, denote the d -th coordinate of \mathbf{v} by v_d . Define the half-lattice

$$\mathbb{L}_+(1) = \{\mathbf{v} \in \mathbb{L} \mid v_d \geq 1\}. \quad (60)$$

Choose the origin in $\mathbb{L}_+(1)$ at $\mathbf{1} = \mathbf{0} + \mathbf{e}_d$ and let P_p^+ be the (usual homogeneous) percolation measure in the half-lattice \mathbb{L}^+ (see for example reference [4]). Since $p > p_c(d)$, with probability 1 there is an infinite cluster C_+ in \mathbb{L}^+ [18], which is unique by the corollary to theorem 1.1 in [4]. Let C be the cluster containing $\mathbf{0}$ in \mathbb{L} . By noting that the edge $\mathbf{0} \sim \mathbf{1}$ is open with probability p , and using the FKG inequality and the fact that $P_p^+(\mathbf{v} \in C_+)$ is an increasing function of v_1 , we see that for any $\mathbf{v} \in \mathbb{L}_+(1)$ we have

$$\begin{aligned} P_{p,\sigma}^I(\mathbf{v} \in C) & \geq p P_p^+(\mathbf{v} \in C(\mathbf{1})) \\ & \geq p P_p^+(\mathbf{v} \in C_+ \text{ and } \mathbf{1} \in C_+) \end{aligned}$$

$$\begin{aligned}
&\geq p P_p^+(\mathbf{v} \in C_+) P_p^+(\mathbf{1} \in C_+) \\
&\geq p \left(P_p^+(\mathbf{1} \in C_+) \right)^2 > 0.
\end{aligned} \tag{61}$$

This is uniform for all $\mathbf{v} \in \mathbb{L}_+(1)$. A similar bound follows if $-\mathbf{v} \in \mathbb{L}_+(1)$. If $\mathbf{v} \in \mathbb{L}_0$, then since the edge $\mathbf{v} \sim (\mathbf{v} + \mathbf{e}_d)$ is open with probability p , we see that

$$P_{p,\sigma}^I(\mathbf{v} \in C) \geq p P_{p,\sigma}^I(\mathbf{v} + \mathbf{e}_d \in C) \geq p^2 \left(P_p^+(\mathbf{1} \in C_+) \right)^2$$

since $\mathbf{v} + \mathbf{e}_d \in \mathbb{L}_+(1)$. Therefore $p^2 \left(P_p^+(\mathbf{1} \in C_+) \right)^2$ is a positive lower bound for $P_{p,\sigma}^I(\mathbf{v} \leftrightarrow \mathbf{0})$ that is uniform in $\mathbf{v} \in \mathbb{L}$.

6 Collapsing animals, and the function $\zeta^I(p, \sigma)$

6.1 Lattice animals, collapse and (homogeneous) percolation

A *lattice animal* is a connected and finite subgraph of \mathbb{L} . All animals will be rooted at the origin, unless otherwise indicated.

The *size* of the animal is its number of vertices, and the *perimeter* of the animal is the collection of lattice edges which are incident with the animal but are not in the animal. The *perimeter size* is the number of edges in the perimeter.

Let $a_n(t)$ denote the number of distinct animals containing the origin, having n edges, and having perimeter size t . For example, in \mathbb{Z}^d , $a_0(2d) = 1$, $a_1(2d+2) = 2d$, and so on.

As before, denote the the cluster at the origin by C , and let $|C|$ denote the number of vertices in C and $\|C\|$ be the number of edges in C . It is known that the limits

$$\zeta^H(p) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log P_p^H(|C|=n) \quad \text{and} \quad \psi^H(p) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log P_p^H(\|C\|=n) \tag{62}$$

exist [17]. Moreover, since $\frac{1}{d}\|C\| \leq |C| \leq \|C\| + 1$ for all clusters C , it follows that $\zeta^H(p) = 0$ if and only if $\psi^H(p) = 0$.

The weight of the open cluster C at the origin in homogeneous percolation is $p^{\|C\|} q^t$ (where $q = 1 - p$). The probability that C has n edges is

$$P_p^H(\|C\|=n) = \sum_{t \geq 0} a_n(t) p^n q^t. \tag{63}$$

This shows that

$$\psi^H(p) = -\log p - \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{t \geq 0} a_n(t) q^t. \tag{64}$$

A *contact* of an animal is a lattice edge that is not in the animal but whose endpoints are both in the animal. Contacts are part of the perimeter of a cluster — they are closed edges with both endpoints in the open cluster.

An edge is in a *cycle* in the open cluster at the origin if the cluster stays connected when the state of the edge is changed to *closed*. In the context of

the lattice animal, an edge is in a cycle if deleting it does not disconnect the animal. The *cyclomatic index* c of a lattice animal is the maximum number of edges which can be deleted without disconnecting the animal.

A model of lattice animals in the *cycle-contact ensemble* is constructed by counting lattice animals with respect to *cyclomatic index* and *contacts* [30]. Hence, let $a_n(c, k)$ be the number of animals containing the origin with n edges, cyclomatic index c , and k contacts. The partition function of the model is

$$Z_n^A(x, y) = \sum_{\substack{c \geq 0 \\ k \geq 0}} a_n(c, k) x^c y^k. \quad (65)$$

The parameters x and y are the *cycle* and *contact activities* (or generating variables) in the model. The free energy of this model is known to exist [15], and is defined by

$$\mathcal{F}^A(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^A(x, y). \quad (66)$$

For animals in \mathbb{Z}^d , we have $2dv = 2n + t + k$ (where v is the number of vertices), while from Euler's relation we get $c = n - v + 1$. Eliminating v from these two relations implies that the cyclomatic index and the number of contacts are related to the perimeter by

$$t = 2d + 2(d-1)n - k - 2dc \quad (67)$$

Hence, write equation (63) as

$$P_p^H(\|C\|=n) = p^n \sum_{\substack{c \geq 0 \\ k \geq 0}} a_n(c, k) q^{2d+2(d-1)n-k-2dc} \quad (68)$$

Comparing the above expression to equation (65) shows that

$$P_p^H(\|C\|=n) = q^{2d} \left(pq^{2(d-1)} \right)^n Z_n^A(q^{-2d}, q^{-1}). \quad (69)$$

Taking logarithms of both sides, dividing by n and letting $n \rightarrow \infty$ gives

$$\psi^H(p) = -2(d-1) \log q - \log p - \mathcal{F}^A(q^{-2d}, q^{-1}). \quad (70)$$

Since $\psi^H(p) = 0$ if $p > p_c(d)$ and $\psi^H(p) > 0$ for $p < p_c(d)$ (see section 1.1), this proves that

$$\mathcal{F}^A(q^{-2d}, q^{-1}) \begin{cases} < -2(d-1) \log q - \log p, & \text{if } p < p_c(d); \\ = -2(d-1) \log q - \log p, & \text{if } p > p_c(d). \end{cases} \quad (71)$$

In particular, $\mathcal{F}(x, y)$ is non-analytic at $p = p_c(d)$ where $x = (1-p)^{-2d}$ and $y = (1-p)^{-1}$, in which case the animals are weighted as critical percolation clusters and the model undergoes a *collapse phase transition* which may be interpreted as a model for gelation of a random medium. In this phase both x and y are large, and the animals are rich in both cycles and contacts, resulting in compact clusters.

6.2 Proof of theorem 4

Our strategy is to bound $P_{p,\sigma}^I(|C|=n)$ from below by $P_p^H(|C|=n-1)$, and then to use the lower bound from homogeneous percolation (see equation (7)).

Let \mathbb{L}_+ be the positive half-lattice, consisting of vertices $\{z \in \mathbb{Z}^d : z_d \geq 1\}$ and all induced edges. Let \mathcal{A}_+ be the set of animals D that are contained in \mathbb{L}_+ and rooted at the vertex $e_d = (0, \dots, 0, 1)$. Then each $D \in \mathcal{A}_+$ is the translation of exactly $\|D\|$ animals which are rooted at the origin in \mathbb{L} , and conversely every animal containing the origin is the translation of at least one animal in \mathcal{A}_+ .

For $n, m, t \geq 0$, let $a[n, m, t]$ (respectively, $a_+[n, m, t]$) be the number of animals rooted at the origin (respectively, the number of animals in \mathcal{A}_+ rooted at e_d) which have n vertices, m edges, and perimeter size t . Then the preceding paragraph shows that $a_+[n, m, t] \geq \frac{1}{n} a[n, m, t]$.

For each $D \in \mathcal{A}_+$, let \hat{D} be the animal in \mathbb{L}_+ obtained by adding the edge $0 \sim e_d$ to D . If D has m edges, and perimeter size t , then \hat{D} has $m+1$ edges (all in $\mathbb{E} \setminus \mathbb{E}_0$) and perimeter size $t+2(d-1)$ with exactly $2s$ perimeter edges in \mathbb{E}_0 .

Thus we have (with $q = 1 - p$)

$$\begin{aligned} P_{p,\sigma}^I(|C| = n) &\geq \sum_{D \in \mathcal{A}_+ : \|D\| = n-1} P_{p,\sigma}^I(C = \hat{D}) \\ &= \sum_{m,t \geq 0} a_+[n-1, m, t] p^{m+1} q^{t+2(d-s-1)} \sigma^{2s} \\ &\geq \frac{1}{n-1} p q^{2(d-s-1)} \sigma^{2s} \sum_{m,t \geq 0} a[n-1, m, t] p^m q^t \\ &= \frac{1}{n-1} p q^{2(d-s-1)} \sigma^{2s} P_p^H(|C| = n-1). \end{aligned}$$

Theorem 4 now follows from equation (7).

6.3 Lattice animals, adsorption and inhomogeneous percolation

In this section our aim is to make a link between inhomogeneous percolation and a model of lattice animals, similar in nature to the association made in section 6.1 for homogeneous percolation.

Our goal is to prove existence of the limits

$$\zeta^I(p, \sigma) = - \lim_{n \rightarrow \infty} \frac{1}{n} P_{p,\sigma}^I(|C|=n) \text{ and } \psi^I(p, \sigma) = - \lim_{n \rightarrow \infty} \frac{1}{n} P_{p,\sigma}^I(\|C\|=n) \quad (72)$$

and to relate these to singular points in the free energies of lattice animals.

We first show existence of the limits in equation (72).

Let \mathcal{A} be the set of lattice animals in \mathbb{L} containing the origin. Let $a_{n,m}(t, r)$ be the number of animals in \mathcal{A} having n edges, of which m are in \mathbb{E}_0 , and whose perimeter consists of t edges in $\mathbb{E} \setminus \mathbb{E}_0$ and r edges in \mathbb{E}_0 .

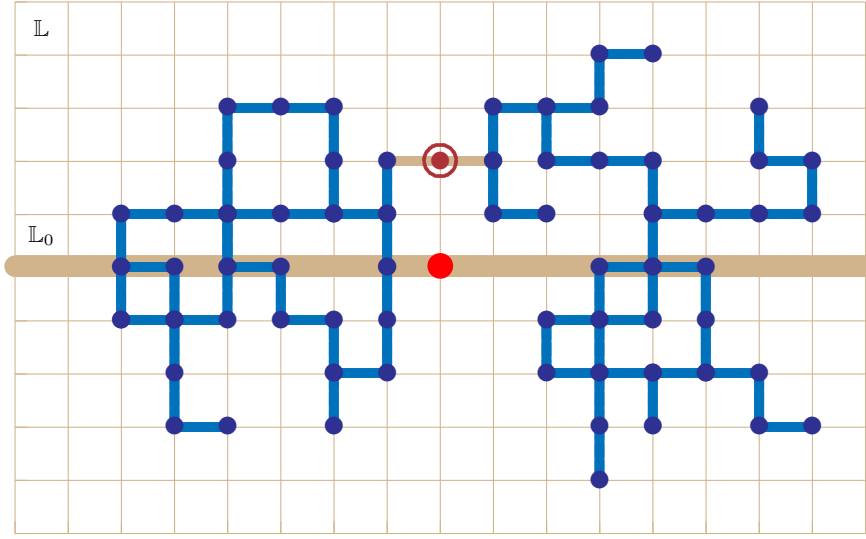


Fig. 7 Concatenation of two clusters in the inhomogeneous lattice. Two animals are placed in a standard placing with a minimal distance of two lattice steps separating them. The animals are joined into a single animal by adding two edges and a single new vertex (marked above) along the lexicographic least path separating the two animals.

Define the partition function of these animals by

$$Z_n^I(x, y, z) = \sum_{\substack{t \geq 0 \\ r \geq 0}} \sum_{m \geq 0} a_{n,m}(t, r) x^t y^r z^m. \quad (73)$$

Then the probability that the cluster at the origin has n edges is given by

$$P_{p,\sigma}^I(\|C\|=n) = p^n Z_n^I(q, \tau, \sigma/p) \quad (74)$$

where $q = 1 - p$ and $\tau = 1 - \sigma$. This shows that if the limiting free energy $\mathcal{F}(x, y, z) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^I(x, y, z)$ exists, then the limit $\psi^I(p, \sigma)$ in equation (72) also exists. Existence of $\zeta^I(p, \sigma)$ is done using a similar approach, but counting animals in a different ensemble (number of vertices).

The basic construction for showing the existence of $\mathcal{F}(x, y, z)$ is illustrated in figure 7. Consider two animals ω_1 and ω_2 , each intersecting \mathbb{L}_0 at vertices we call *visits*. An *edge-visit* in these animals is an edge of the animal which is also in \mathbb{E}_0 . Observe that translations parallel to \mathbb{L}_0 preserve visits and edge-visits.

The goal is to concatenate ω_1 and ω_2 into one animal from which the original pair of animals can be uniquely recovered.

A *placing* $(\hat{\omega}_1, \hat{\omega}_2)$ of two animals ω_1 and ω_2 is a pair of translations (parallel to \mathbb{L}_0) $\hat{\omega}_1$ of ω_1 and $\hat{\omega}_2$ of ω_2 such that the minimum distance between $\hat{\omega}_1$ and $\hat{\omega}_2$ is at least 2 steps.

There are infinitely many placings $(\hat{\omega}_1, \hat{\omega}_2)$, but there are only finitely many non-equivalent placings with a minimum distance of two (where two

placings are equivalent if they only differ by an overall translation parallel to \mathbb{L}_0).

Consider a placing $(\hat{\omega}_1, \hat{\omega}_2)$ with the following properties: (1) each visit in $\hat{\omega}_1$ is lexicographically less than each visit in $\hat{\omega}_2$; (2) the shortest path in \mathbb{L} from a vertex in $\hat{\omega}_1$ to a vertex in $\hat{\omega}_2$ has length two. These two properties define a nonempty finite collection of placings (up to equivalence), one of which is lexicographically least. This is the *standard placing*.

Observe that the total perimeter of the animals in a standard placing is the sum of the perimeters of the two animals.

In each standard placing there is at least one path of length two joining the two animals. In the set of such paths, there is a path P which is lexicographically least. The animals $\hat{\omega}_1$ and $\hat{\omega}_2$ may be concatenated by joining them into a single animal by adding two edges along P . This increases the number of edges by 2 and decreases the total perimeter of the animals by 2. Observe that the center vertex of P is a cut-vertex in the concatenated animal.

Consider the possible arrangements of the two added edges along P : (a) The two added edges are disjoint with \mathbb{L}_0 . (b) One edge in P is in \mathbb{L}_0 , and (c) both edges are in \mathbb{L}_0 .

Next, account for the change in the perimeter of the animals upon concatenation. Suppose that ω_1 is an animal with n_1 edges and with $m - m_1$ edges in \mathbb{L}_0 , and with perimeter size $t + r - (t_1 + r_1)$, including $r - r_1$ perimeter edges in \mathbb{L}_0 .

Similarly, suppose that ω_2 is an animal with n_2 edges and with m_1 edges in \mathbb{L}_0 , and with perimeter size $t_1 + r_1$, including r_1 perimeter edges in \mathbb{L}_0 .

Putting these animals in a standard placing and concatenating them gives an animal ω with $n_1 + n_2 + 2$ edges in total, and there are either m edge-visits (case (a)), or $(m + 1)$ edge-visits (case (b)), or $(m + 2)$ edge-visits (case (c)). These different outcomes are due to the fact that new edges may be created in \mathbb{L}_0 when the concatenation introduces two new edges.

It is necessary that ω_1 and ω_2 can be recovered from the concatenated animal. Since the concatenation is done by adding two edges incident with one another in a new cut-vertex, these edges can be located in ω by colouring the new vertex red. This gives an animal with one red vertex of degree 2 (and the remaining vertices are all black). Note that the maximum number of vertices in ω is $n_1 + n_2 + 3$.

By deleting the two edges incident on the red vertex, it is possible to recover the two translated animals $\hat{\omega}_1$ and $\hat{\omega}_2$ in their standard placing. Observe that there are at most $2d - 2$ new perimeter edges associated with the red vertex, and that at most $2s$ of these may be in the defect lattice \mathbb{L}_0 .

We now account for the changes in the number of perimeter edges. The concatenation deletes two perimeter edges, but the new red vertex creates new perimeter edges. Thus, ω has perimeter between $t + r - 2$ and $t + r - 2 + 2d - 2$ of which between r and $r + 2s$ are in \mathbb{L}_0 .

The roots of the animals ω_i are discarded when they are put in standard placing, and so the number of choices for each $\hat{\omega}_i$ is at least $a_{n_1, m - m_1}(t - t_1, r - r_1)/(n_1 + 1)$ for $\hat{\omega}_1$ and $a_{n_2, m_1}(t_1, r_1)/(n_2 + 1)$ for $\hat{\omega}_2$.

The concatenated animal ω is similarly unrooted, and there are at most $(n_1 + n_2 + 3)$ positions for the red vertex. Accounting for the different possible numbers of edge-visits and perimeter sizes then shows that

$$\begin{aligned} & \sum_{m_1, t_1, r_1} \left(\frac{a_{n_1, m_1}(t-t_1, r-r_1)}{n_1+1} \right) \left(\frac{a_{n_2, m_1}(t_1, r_1)}{n_2+1} \right) \\ & \leq (n_1 + n_2 + 3) \sum_{i=0}^{2d-2} \sum_{j=0}^{2s} [a_{n_1+n_2+2, m}(t-2+i, r+j) \\ & + a_{n_1+n_2+2, m+1}(t-1+i, r-1+j) + a_{n_1+n_2+2, m+2}(t+i, r-2-j)], \end{aligned} \quad (75)$$

where the summation over i and j accounts for new perimeter edges incident on the red vertex.

Define $\phi(x, y) = \sum_{i=0}^{2d-2} \sum_{j=0}^{2s} x^{-i} y^{-j}$. Multiply equation (75) by $x^t y^r z^m$ and sum over m, t and r . This gives

$$\begin{aligned} Z_{n_1}(x, y, z) Z_{n_2}(x, y, z) & \leq (n_1 + n_2 + 1)^2 (n_1 + n_2 + 3) \phi(x, y) \\ & \times [x^2 + z^{-1}xy + z^{-2}y^2] Z_{n_1+n_2+2}(x, y, z). \end{aligned} \quad (76)$$

Define $\lambda(x, y, z) = \phi(x, y) (x^2 + z^{-1}xy + z^{-2}y^2)$. Then the above simplifies to

$$\begin{aligned} Z_{n_1}(x, y, z) Z_{n_2}(x, y, z) & \leq (n_1 + n_2 + 1)^2 (n_1 + n_2 + 3) \\ & \times \lambda(x, y, z) Z_{n_1+n_2+2}(x, y, z). \end{aligned} \quad (77)$$

This shows that the function $Z_{n-2}(x, y, z)/\lambda(x, y, z)$ satisfies a generalised supermultiplicative inequality on \mathbb{N} , and by references [20, 23] one obtains the following theorem.

Theorem 6 *For $x, y, z \in (0, \infty)$ the limit*

$$\mathcal{F}^I(x, y, z) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^I(x, y, z)$$

exists. Moreover, $\mathcal{F}^I(x, y, z)$ is log-convex in each of its arguments. \square

Log-convexity follows because $Z_n^I(x, y, z)$ is a polynomial in $\{x, y, z\}$ with positive coefficients.

Comparison to equation (74) gives the following relationship between $\mathcal{F}^I(x, y, z)$ and $\zeta^I(p, \sigma)$:

$$\mathcal{F}^I(q, \tau, \sigma/p) = -\log p - \psi^I(p, \sigma) \quad (78)$$

which is valid for $p, \sigma \in (0, 1)$ and proves existence of the limit definition of $\psi^I(p, \sigma)$ in equation (72).

Existence of $\zeta^I(p, \sigma)$ can be similarly shown, as follows.

Let $A_{v, n, m}(t, r)$ be the number of edge animals at the origin as above, but with v vertices, n edges of which m are in \mathbb{E}_0 , and with perimeter having t edges in $\mathbb{E} \setminus \mathbb{E}_0$ and r edges in \mathbb{E}_0 . Define the partition function

$$Y_v(a, x, y, z) = \sum_{\substack{t \geq 0 \\ r \geq 0}} \sum_{\substack{n \geq 0 \\ m \geq 0}} A_{v, n, m}(t, r) a^n x^t y^r z^m. \quad (79)$$

Then the probability that the animal at the origin has size v is given by

$$P_{p,\sigma}^I(|C| = v) = Y_v(p, q, \tau, \sigma/p). \quad (80)$$

Repeating the construction in figure 7 in this ensemble gives an outcome similar to the above, but now with

$$\begin{aligned} & \sum_{n_1, m_1, t_1, r_1} \left(\frac{A_{v_1, n-n_1, m-m_1}(t-t_1, r-r_1)}{v_1} \right) \left(\frac{A_{v_2, n_1, m_1}(t_1, r_1)}{v_2} \right) \\ & \leq (v_1 + v_2 + 1) \sum_{i=0}^{2d-2} \sum_{j=0}^{2s} [A_{v_1+v_2+1, n+2, m}(t-2+i, r+j) \\ & \quad + A_{v_1+v_2+1, n+2, m+1}(t-1+i, r-1+j) + A_{v_1+v_2+1, n+2, m+2}(t+i, r-2+j)]. \end{aligned} \quad (81)$$

Multiply this by $a^n x^t y^r z^m$ and summing the left hand side over $\{n, t, r, m\}$ gives

$$Y_{v_1}(a, x, y, z) Y_{v_2}(a, x, y, z) \leq v_1 v_2 (v_1 + v_2 + 1) [a^{-2} \phi(x, y) (x^2 + xyz^{-1} + y^2 z^{-2})] Y_{v_1+v_2+1}(a, x, y, z). \quad (82)$$

Similarly to theorem 6, $Y_v(a, x, y, z)$ satisfies a generalised supermultiplicative inequality on \mathbb{N} , and by references [20, 23] the following theorem is a result.

Theorem 7 For $a, x, y, z \in (0, \infty)$ the limit

$$\mathcal{G}(a, x, y, z) = \lim_{v \rightarrow \infty} \frac{1}{v} \log Y_v(a, x, y, z)$$

exists. Moreover, $\mathcal{G}(a, x, y, z)$ is log-convex in each of its arguments. \square

Notice by equation (80) that $\zeta^I(p, \sigma) = -\mathcal{G}(p, q, \tau, \sigma/p)$ so that the limit in equation (72) exists.

We claim that $\zeta^I(p, \sigma) = 0$ in \mathcal{R}_H . To see this, suppose $\zeta^I(p, \sigma) > 0$ at some (p, σ) in \mathcal{R}_H . Then there exists an $\epsilon > 0$ and an $N_\epsilon \in \mathbb{N}$ such that $P_{p,\sigma}^I(|C| = n) \leq e^{-\epsilon n}$ for all $n \geq N_\epsilon$. This shows that

$$P_{p,\sigma}^I(\infty > |C| \geq n) \leq \sum_{m \geq n} e^{-\epsilon m} = \frac{e^{-\epsilon n}}{1 - e^{-\epsilon}}$$

for any $n \geq N_\epsilon$. But this contradicts theorem 4. Therefore $\zeta^I(p, \sigma) = 0$.

A similar argument using theorem 5 shows that $\zeta^I(p, \sigma) = 0$ in \mathcal{R}_L .

Since $\zeta^I(p, \sigma) = 0$ in $\mathcal{R}_H \cup \mathcal{R}_L$, it follows that $\psi^I(p, \sigma) = 0$ in $\mathcal{R}_H \cup \mathcal{R}_L$.

On the other hand, by theorem 3, $\zeta^I(p, \sigma) \geq \frac{1}{2\chi^H(p)\chi^I(p, \sigma)} > 0$, provided that $p < p_c(d)$ and $\sigma \in (0, \sigma^*(p))$ (this is in regime \mathcal{R}_0 in figure 3). This shows that $\psi^I(p, \sigma) > 0$ in \mathcal{R}_0 .

In terms of the free energy $\mathcal{F}^I(x, y, z)$ in equation (78), this implies that

$$\mathcal{F}^I(q, \tau, \sigma/p) \begin{cases} = -\log p, & \text{in } \mathcal{R}_H \text{ (i.e. when } p > p_c(d)); \\ = -\log p, & \text{in } \mathcal{R}_L \text{ (when } p < p_c(d) \text{ and } \sigma > \sigma^*(p)); \\ < -\log p, & \text{in } \mathcal{R}_0 \text{ (when } p < p_c(d) \text{ and } \sigma < \sigma^*(p)). \end{cases} \quad (83)$$

Thus, $\mathcal{F}^I(q, \tau, \sigma/p)$ is non-analytic along the line segment $p = p_c(d)$ and $\sigma \in (0, \sigma^{**})$ (where σ^{**} is the limit of $\sigma^*(p)$ as p approaches $p_c(d)$ from the left), as well as along the surface critical curve $\sigma^*(p)$ for $0 \leq p < p_c(d)$.

7 Numerical results

We performed a numerical study of inhomogeneous percolation using the Newman-Ziff algorithm [34] to sample clusters in the model. To describe the implementation, let $B(L)$ be the d -dimensional hypercube of side length $2L$ defined by $B(L) = [-L, L]^d \cap \mathbb{L}$. The boundary of $B(L)$ is $\partial B(L)$, and it has a vertical and a horizontal component, similar to equation (50):

$$\partial_{\text{vert}}(L) = \{v \in B(L) \mid |v_i| = L \text{ for at least one } i \leq s\} \quad (84)$$

$$\partial_{\text{hor}}(L) = \{v \in B(L) \mid |v_i| = L \text{ for at least one } i > s\}. \quad (85)$$

The vertical component $\partial_{\text{vert}}(L)$ is composed of $2s$ $(d-1)$ -dimensional hypercubes defined by

$$A_i = \{v \in \partial_{\text{vert}}(L) \mid v_i = L\} \text{ and } A_{-i} = \{v \in \partial_{\text{vert}}(L) \mid v_i = -L\}$$

for $i = 1, 2, \dots, s$.

Consider a realisation of open edges in \mathbb{L} at densities (p, σ) . This realisation gives sets of open edges in $B(L)$: Denote the set of open edges in $B(L) \setminus \mathbb{L}_0$ by \mathcal{P} and the set of open edges in the defect plane $B(L) \cap \mathbb{L}_0$ by \mathcal{S} . Let $\mathbb{1}_{\mathcal{P}, \mathcal{S}, L}(A_1 \rightsquigarrow A_{-1})$ be the indicator function that there is an open path inside $B(L)$ between two opposite vertical faces A_1 and A_{-1} in $\partial_{\text{vert}}(L)$.

The average of $\mathbb{1}_{\mathcal{P}, \mathcal{S}, L}(A_1 \rightsquigarrow A_{-1})$ for all realisations of \mathcal{P} at density p and all \mathcal{S} with $|\mathcal{S}| = s$ is denoted by $Q_p(s)$. That is, $Q_p(s)$ is the probability that there is an open path in $B(L)$ between A_1 and A_{-1} when bulk edges are open with probability p given that there are exactly s surface edges open in $B(L) \cap \mathbb{L}_0$. Then $0 \leq s \leq S$ where S is the total number of edges in $B(L) \cap \mathbb{L}_0$.

Following Newman and Ziff [34], let us construct

$$Q_L(p, \sigma) = \sum_{s=0}^S \binom{S}{s} \sigma^s (1 - \sigma)^{S-s} Q_p(s). \quad (86)$$

Clearly, $Q_L(p, \sigma)$ decreases to zero with L in \mathcal{R}_0 (see figure 3). On the other hand, it should approach a positive probability with increasing σ for fixed $p \leq p_c(d)$ when $\sigma > \sigma^*(p)$. That is, if $L \rightarrow \infty$, then $\lim_{L \rightarrow \infty} Q_L(p, \sigma) = 0$ in \mathcal{R}_0 , and $\liminf_{L \rightarrow \infty} Q_L(p, \sigma) > 0$ in the surface regime \mathcal{R}_L . Hence, one may estimate the critical curve $\sigma^*(p)$ by estimating $Q_L(p, \sigma)$ for finite values of L and various (p, σ) .

In figure 8 numerical estimates of $Q_L(\sigma) \equiv Q_L(\sigma, p)$ as a function of σ for $p = 0.1$ for the model with $(d, s) = (3, 2)$ is presented (with $L \in \mathcal{L} = \{5, 10, 15, 20, 25, 30, 35\}$). $Q_L(\sigma)$ is small for σ small, and increases with increasing σ . For different values of L all the curves pass almost through the same point at a critical value of σ .

The normal scaling assumption for a function like $Q_L(\sigma)$ is

$$Q_L(\sigma) \simeq F_p(L^\phi(\sigma - \sigma^*(p))) \quad (87)$$

where ϕ is a crossover exponent and F is a scaling function. In the case that $s = 2$ the surface percolation at $\sigma = \sigma^*(p)$ should be in the same universality

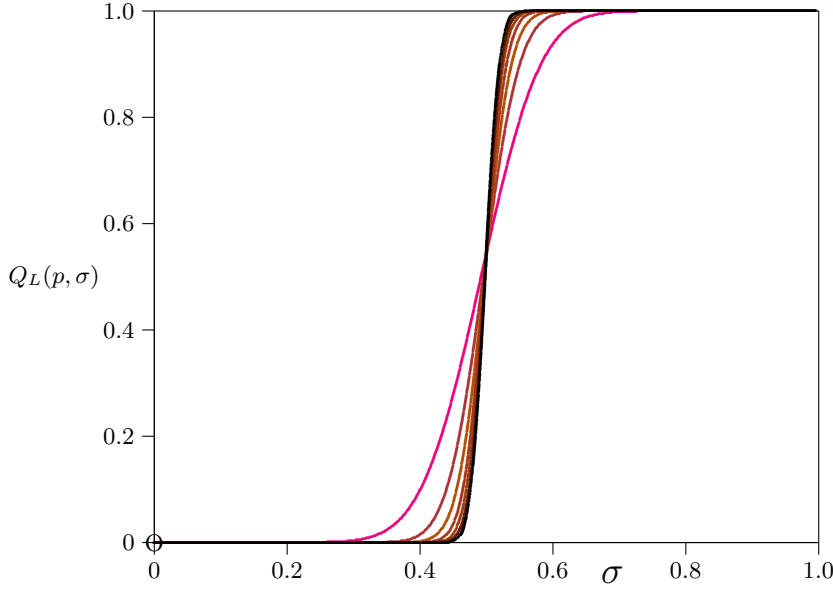


Fig. 8 Plots of $Q_L(p, \sigma)$ for $p = 0.1$ as a function of σ for $d = 3$ and $s = 2$. Each curve is the average taken over 30000 realisations of bulk percolation clusters with $p = 0.1$ and then determining $Q_L(0.1, \sigma)$ as a function of σ . The value of L increases in the set $\{5, 10, 15, 20, 25, 30, 35\}$. The curves intersect to high accuracy in a single point for larger values of L , which gives an estimate of the location of the critical point $\sigma^*(0.1)$.

class as homogeneous percolation in two dimensions. If $\sigma = \sigma^*(p)$ then this shows that $Q_L(\sigma^*(p)) \simeq F_p(0)$ so that the value of $Q_L(\sigma)$ is independent of L at the critical point. This indicates that the point where all the curves intersect in figure 8 is an estimate of the location of the critical point.

To find a numerical estimate of the crossing point, define the least square width of the set of curves at surface density σ by

$$\mathcal{E}^2(\sigma) = \sum_{L \in \mathcal{L}} \sum_{K \in \mathcal{L}} (Q_L(\sigma) - Q_K(\sigma))^2. \quad (88)$$

$\mathcal{E}^2(\sigma)$ is a measure of the square vertical width of the set of curves, and minimizing it gives an estimate of the location (the value of σ) of *narrowest vertical waist* in the set of intersecting curves. That is, this gives an estimate for $\sigma^*(p)$. An error bar can be estimated by determining the values of σ where $\mathcal{E}^2(\sigma)$ is twice its minimum. For example, the data in figure 8 gives $\sigma^*(0.1) = 0.49859 \pm 0.00040$. A plot of $\mathcal{E}^2(\sigma)$ against σ for $p = 0.1$ is given in figure 9.

If the data at $L = 5$ are dropped, then a similar analysis show that $\sigma^*(0.1) = 0.49879 \pm 0.00059$. Similarly, dropping both $L = 5$ and $L = 10$ from the analysis gives $\sigma^*(0.1) = 0.499081 \pm 0.00050$. Comparing these results show that there is no improvement in the statistical estimate by dropping data at small values of L , and so we take as our best estimate the result when dropping $L = 5$ from the analysis, namely $\sigma^*(0.1) = 0.49879 \pm 0.00059$.

The curves in figure 9 show a systematic drift towards the right with removing data at the smallest values of L . We estimate a systematic error in the data by taking twice the absolute difference between the estimate over all the data and the estimate with the data at $L = 5$ removed. This gives $\sigma^*(0.1) = 0.49879 \pm 0.00059 \pm 0.00042$ where the last error bar is an estimated systematic error.

By adding the two error bars our best estimate is obtained, $\sigma^*(0.1) = 0.4988 \pm 0.0011$

A similar approach at $p = 0$ gives the results $\sigma^*(0) = 0.49986 \pm 0.00026$ over all the data and $\sigma^*(0) = 0.50003 \pm 0.00034$ if the data at $L = 5$ is ignored. This gives our best estimate $\sigma^*(0) = 0.50003 \pm 0.00034 \pm 0.00034$ so that by combining the error bars, $\sigma^*(0) = 0.50003 \pm 0.00068$ (consistent with the exact value for bond percolation in the square lattice [22, 24]; see proposition 1(b)).

Similar analysis can be done at other values of p and the results are tabulated in table 7. The stated error bar is the sum of the statistical and systematic error. In figure 11 the results are plotted in the (p, σ) -plane. The critical curve varies slowly with p for small p , but decreases quickly for p approaching $p_c(3)$.

An interesting situation arises when $p = p_c(3)$. Simulations for $d = 3$ and $s = 2$ can be done with $p = 0.24881182 = p_3^* \approx p_c(3)$, very close to the critical point (the uncertainty is only in the last digit) for percolation in the cubic lattice ($d = 3$) [42], see reference [28]. In figure 10 estimates of $Q_L(\sigma, p_c(d))$ are plotted against σ for L taking values in $\{5, 10, 15, 20, 25, 30, 35\}$ for $d = 3$ and $s = 2$.

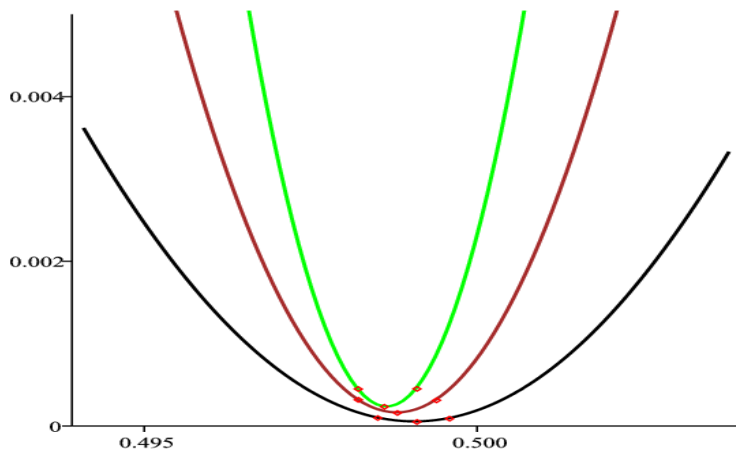


Fig. 9 A plot of $\mathcal{E}^2(\sigma)$ against σ for $p = 0.1$, $d = 3$ and $s = 2$. The top curve is for data when $\mathcal{L} = \{5, 10, 15, 20, 25, 30\}$, the middle curve is obtained with $L = 5$ dropped, and the bottom curve is obtained when both $L = 5$ and $L = 10$ are dropped. The minima in these curves are estimates of the location of the narrow point in figure 8. The width of $\mathcal{E}^2(\sigma)$ at twice its minimum height, and also at four times its minimum height, is indicated by the square symbols on each curve.

$d = 3, s = 2$		$d = 4, s = 2$	
p	$\sigma^*(p)$	p	$\sigma^*(p)$
0.000	0.5	0	0.5
0.050	0.50043 ± 0.00064	0.050	0.5007 ± 0.0013
0.100	0.4988 ± 0.0011	0.0750	0.4992 ± 0.0010
0.150	0.4929 ± 0.0018	0.1100	0.4956 ± 0.0010
0.170	0.4865 ± 0.0020	0.1200	0.4929 ± 0.0019
0.200	0.4746 ± 0.0024	0.1250	0.4914 ± 0.0014
0.215	0.4626 ± 0.0026	0.1300	0.4896 ± 0.0019
0.220	0.4575 ± 0.0020	0.1350	0.4873 ± 0.0026
0.225	0.4509 ± 0.0010	0.1400	0.4842 ± 0.0036
0.230	0.4405 ± 0.0021	0.1450	0.4796 ± 0.0063
0.235	0.4308 ± 0.0019	0.1500	0.4741 ± 0.0052
0.240	0.4138 ± 0.0046	0.1525	0.4703 ± 0.0030
0.245	0.3797 ± 0.0080	0.1550	0.4651 ± 0.0040
p_3^*	0.2941 ± 0.0091	0.1575	0.4551 ± 0.0025
		0.1585	0.4463 ± 0.0032
		0.1595	0.4290 ± 0.0045
		0.1600	0.4079 ± 0.0062
		0.16013	0.3977 ± 0.0056

Table 1 Estimates of $\sigma^*(p)$ for $(d, s) = (3, 2)$ and $(d, s) = (4, 2)$

Minimizing $\mathcal{E}^2(\sigma)$ over all the data gives $\sigma^*(p_3^*) = 0.2949 \pm 0.0053$ and if the data point at $L = 5$ is dropped, then $\sigma^*(p_3^*) = 0.2941 \pm 0.0075$. This gives the best estimate $\sigma^*(p_3^*) = 0.2941 \pm 0.0075 \pm 0.0016$. Combining the error bars give $\sigma^*(p_3^*) = 0.2941 \pm 0.0091$.

Notice that the numerical result for $\sigma^*(p_3^*)$ rules out the critical bulk percolation density at $p_c(3) = 0.24881182(2)$ in its error bars. Since we do not know that $\sigma^*(p)$ is left-continuous at $p = p_c(3)$ this result cannot be interpreted as evidence that $\lim_{p \rightarrow p_c(3)^-} \sigma^*(p) > \sigma^*(p_c(3))$ – this is so in particular also because of the steepness of $\sigma^*(p)$ as p approaches $p_c(3)$ from below, as seen in figure 11.

Numerical simulations of the model with $d = 4$ and $s = 2$ were also done for $L \in \{5, 8, 10, 12, 15, 20, 25, 30\}$ and a select set of values of the bulk density p approaching $p_c(4)$. In the case that $p = 0.05$ a plot of $Q_L(\sigma)$ against σ is similar to figure 8. Minimizing $\mathcal{E}^2(\sigma)$ gives an estimate of the critical point by locating the narrowest waist in the set of crossing curves. Over all the data this gives $\sigma^*(0.05) = 0.50043 \pm 0.00070$ and if the data point at $L = 5$ is dropped, $\sigma^*(0.05) = 0.50070 \pm 0.00067$. Computing a systematic error as before by doubling the absolute difference between the estimates gives a best estimate of $\sigma^*(0.05) = 0.50070 \pm 0.00067 \pm 0.00054$ and combining the error bars gives the $\sigma^*(0.05) = 0.5007 \pm 0.0013$.

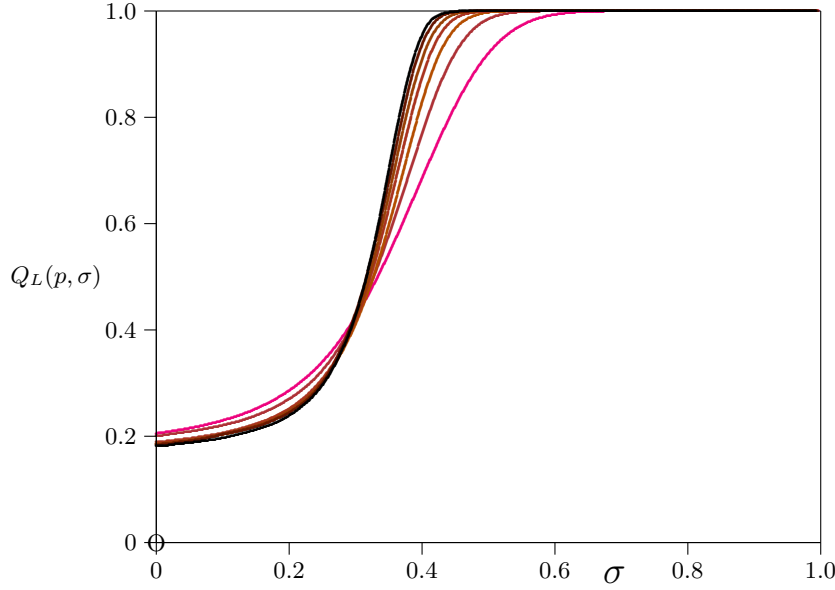


Fig. 10 Critical behaviour in the model for $p = p_c(3) = 0.24881182(10)$ as a function of σ for $d = 3$ and $s = 2$. The curves are numerical estimates of the probability $Q_L(p, \sigma)$, for $L \in \{5, 10, 15, 20, 25, 30, 35\}$. Each curve was computed by taking the average of 30000 realisations of bulk percolation clusters at $p = p_c(3)$ and then computing the average $Q_L(p, \sigma)$ as a function of σ . The intersections between the curves is a numerical estimate of the location of the critical point $\sigma^*(p_c(3))$.

The estimates in table 7 for other values of $p \in (0, p_c(4))$ were similarly estimated. In each case $Q_L(\sigma)$ was computed over 30000 realisations of bulk clusters.

We have also performed simulations at $p_4^* = 0.16013$ which is the best estimate for the critical point $p_c(4)$ [36]. Minimizing $\mathcal{E}^2(\sigma)$ over all the data gives $\sigma^*(p_4^*) = 0.3962 \pm 0.0031$ and if the data point at $L = 5$ is dropped, then $\sigma^*(p_4^*) = 0.3977 \pm 0.0025$. This gives the best estimate $\sigma^*(p_3^*) = 0.3977 \pm 0.0025 \pm 0.0031$. Combining the error bars give $\sigma^*(p_4^*) = 0.2941 \pm 0.0056$.

The critical curve $\sigma^*(p)$ against p for $(d, s) = (4, 2)$ is plotted in figure 13.

Similar to the case for $(d, s) = (3, 2)$ the numerical data for $(d, s) = (4, 2)$ suggest that $\sigma^*(p_4^*) > p_c(4) \approx 0.160130 \pm 0.000003$ [36]. The estimate at p_4^* is far larger than $p_c(4)$, but as above this cannot be interpreted as evidence that $\lim_{p \rightarrow p_c(4)^-} \sigma^*(p) > \sigma^*(p_c(4))$.

8 Conclusions

In this paper we have generalised homogeneous percolation in \mathbb{L} to a model of inhomogeneous percolation in a d -dimensional \mathbb{L} with an s -dimensional defect plane. We showed that there is a surface transition in this model,

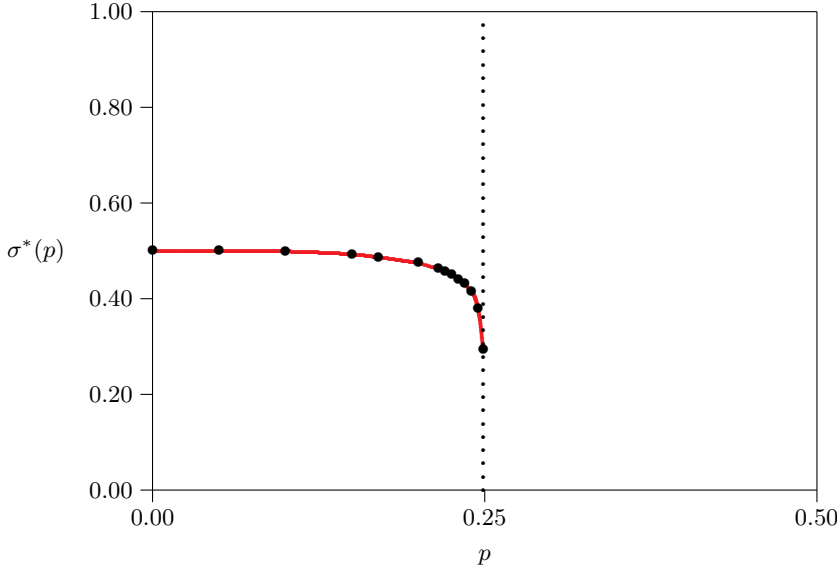


Fig. 11 Numerical estimates of the location of the critical curve $\sigma^*(p)$ as a function of p for $(d, s) = (3, 2)$ are indicated by the data points (\bullet). The solid curve is an interpolation curve drawn through the data points.

as proposed by references [6, 8, 10, 11, 12]. There is a critical curve $\sigma^*(p)$ for $p \in [0, 1]$, with the properties that $\sigma^*(p) > p_c(d) > 0$ for $p < p_c(d)$ while $\sigma^*(p) = 0$ if $p > p_c(d)$, and that σ^* is a strictly decreasing function of p for $p < p_c(d)$ (see propositions 1 and 2). It follows that σ^* is discontinuous at $p = p_c(d)$. We expect that σ^* is continuous for $p < p_c(d)$, but we have not yet proven this.

We have also examined the nature of the surface transition in this model. We investigated the three phases: the subcritical phase \mathcal{R}_0 in which all clusters are finite, the surface supercritical phase \mathcal{R}_L in which the infinite cluster stays near the defect surface, and the bulk supercritical phase \mathcal{R}_H in which the infinite cluster permeates the whole lattice. We generalised the differential inequalities of homogeneous percolation [1] to the model here and showed (theorem 1) that the susceptibility $\chi^I(p, \sigma)$ is infinite if and only if $\theta^I(p + \delta, \sigma + \delta) > 0$ for all small $\delta > 0$ (which happens whenever (p, σ) is not in the interior of the subcritical phase \mathcal{R}_0).

In section 4 we considered the exponential decay of the cluster size distribution in the subcritical phase. We show that the cluster size distribution decays exponentially (see theorem 3) in the subcritical phase (i.e., when $p < p_c(d)$ and $\sigma < \sigma^*(p)$). In contrast, theorems 4 and 5 prove subexponential decay of the cluster size distribution in the supercritical phases. Our lower bound for $P_{p, \sigma}^I(|C| = n)$ in \mathcal{R}_H is $\exp(-cn^{(d-1)/d})$, the same as for supercritical homogeneous percolation. However, in \mathcal{R}_L , where the infinite cluster stays close to the defect plane and looks s -dimensional, our lower bound (neglecting a logarithmic term) is $\exp(-cn^{(s-1)/s})$. We expect that these lower bounds are essentially optimal in both supercritical phases, al-

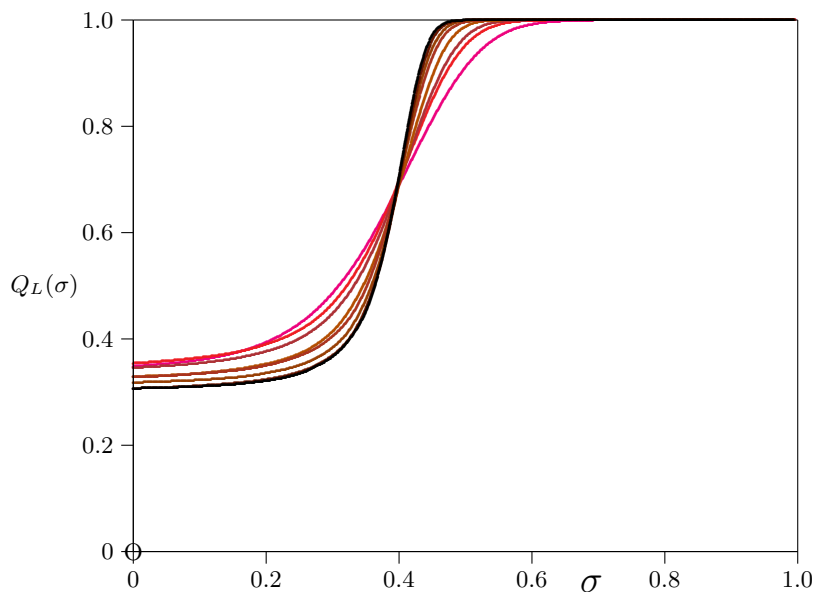


Fig. 12 Critical behaviour in the model for $p = p_c(4) = 0.160130$ as a function of σ for $d = 4$ and $s = 2$. The curves are numerical estimates of the probability $Q_L(p, \sigma)$, for $L \in \{5, 8, 10, 15, 20, 25, 30, 35\}$. Each curve was computed by taking the average of 20000 realisations of bulk percolation clusters at critical density $p_c(4)$ then determining $Q_L(p, \sigma)$ as a function of σ . The intersections of the curves for larger L is a numerical signal of critical behaviour in this model.

though we have not attempted to prove the corresponding (more challenging) upper bounds.

We examined $\zeta^I(p, \sigma)$ and $\psi^I(p, \sigma)$, the exponential decay rates of the cluster size distribution (where “size” is measured by vertices for ζ^I and by edges for ψ^I). We related these functions to the free energy of a model of collapsing lattice animals interacting with a defect plane. We showed that the existence of the free energy in the animal model implies the existence of ζ^I and ψ^I , and we showed that the percolation transition had implications about non analyticity of the free energy.

Finally we performed a numerical study of inhomogeneous percolation using the Newman-Ziff algorithm. We plotted the box crossing probability $Q_L(p, \sigma)$ as a function of σ for various values of $p \in [0, p_c(d)]$. Table 7 shows these results for $(d, s) = (3, 2)$ and for $(d, s) = (4, 2)$, and includes an error bar associated with each estimated $\sigma^*(p)$ value.

For both $d = 3$ and $d = 4$ we find qualitatively similar phase boundaries. In both cases the curves start at $\sigma_c(0) = \frac{1}{2}$ and decreases with increasing p . On approach to $p_c(d)$ the critical curve becomes sensitive to even small changes in p , and $\sigma^*(p)$ is discontinuous at $p = p_c(d)$. There are numerous open question about the function $\sigma^*(p)$, regarding its continuity and rate of decrease with increasing $p < p_c(d)$.

Acknowledgements

EJJvR and NM acknowledge support in the form of NSERC Discovery Grants from the Government of Canada. We also thank Geoffrey Grimmett for some helpful email correspondence.

A Differential Inequalities for Inhomogeneous Percolation

In this appendix our aim is to prove the following differential inequalities

$$\mathbf{q} \cdot \nabla \theta^I(p, \sigma, \gamma) \leq 2d \chi^H(p) \theta^I(p, \sigma, \gamma) (1 - \gamma) \frac{\partial \theta^I}{\partial \gamma}, \quad (89)$$

$$\theta^I(p, \sigma, \gamma) \leq \gamma \frac{\partial \theta^I}{\partial \gamma} + \left(\theta^I(p, \sigma, \gamma) \right)^2 + \chi^H(p) \theta^I(p, \sigma, \gamma) \left(\mathbf{p} \cdot \nabla \theta^I(p, \sigma, \gamma) \right) \quad (90)$$

where $\nabla = (\frac{\partial}{\partial p}, \frac{\partial}{\partial \sigma})$ and $p \leq \sigma$. These are equations (29) and (30).

These inequalities are due to Aizenman and Barsky [1] for homogeneous bond percolation (the proofs can be found in reference [17] as well), and we adapt their proofs to the inhomogeneous model.

Let $B(N)$ be the box of side-length $2N$ centered at the origin of \mathbb{L} . Denote by $V(N)$ the vertices in $B(N)$. We shall prove the above differential inequalities in $B(N)$, and then take $N \rightarrow \infty$.

Impose periodic boundary conditions on $B(N)$ by identifying its opposite faces. Denote this finite lattice by $\mathbb{L}(N)$. Denote the intersection of $\mathbb{L}(N)$ and \mathbb{L}_0 by $\mathbb{L}_0(N)$. Denote the set of edges in $\mathbb{L}_0(N)$ by $\mathbb{E}_0(N)$ and the set of edges in $\mathbb{L}(N)$ by $\mathbb{E}(N)$.

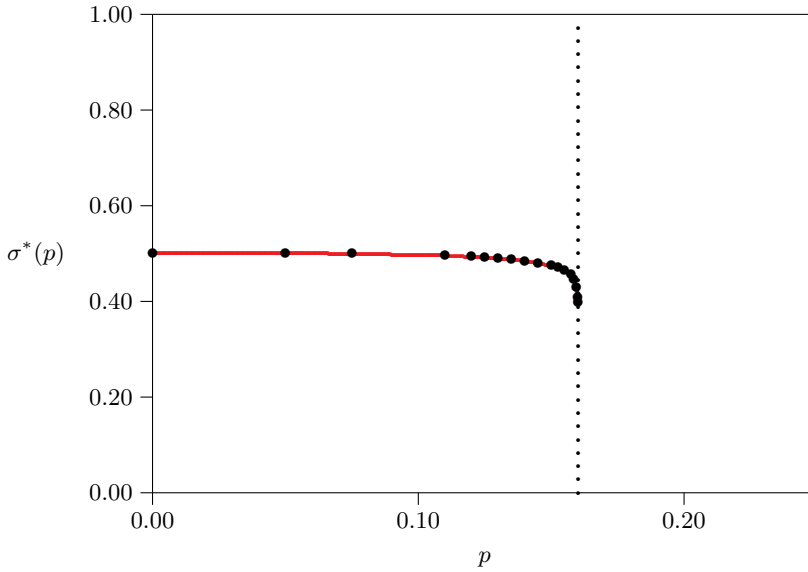


Fig. 13 Numerical estimates of the location of the critical curve $\sigma^*(p)$ as a function of p for $(d, s) = (4, 2)$ are indicated by the data points (\bullet). The solid curve is an interpolation curve drawn through the data points.

As before, edges in $\mathbb{E}(N) \setminus \mathbb{E}_0(N)$ are open with the bulk probability p , and edges in $\mathbb{E}_0(N)$ are open with surface probability σ .

The open cluster in $\mathbb{L}(N)$ at the vertex x is $C_N(x)$, and the open cluster at the origin is $C_N(0) \equiv C_N$. For $y \in V(N)$, define the susceptibility $\chi_N^I(p, \sigma; y) = E_{p, \sigma}^I |C_N(y)|$.

Introduce the ghost vertex g and edges $\mathbb{E}_g(N) = \{g \sim x \mid x \in V(N)\}$. Edges in $\mathbb{E}_g(N)$ are open with probability $\gamma \in (0, 1)$. Define G_N to be the collection of vertices in $\mathbb{L}(N)$ adjacent to g through open edges in $\mathbb{E}_g(N)$.

For inhomogeneous percolation on $\mathbb{E}(N) \cup \mathbb{E}_g(N)$ with parameters (p, σ, γ) , let $P_{p, \sigma, \gamma}^I$ and $E_{p, \sigma, \gamma}^I$ be the corresponding probability measure and expectation. For $y \in \mathbb{L}(N)$, we define the associated quantities

$$\theta_N^I(p, \sigma, \gamma; y) = P_{p, \sigma, \gamma}^I(C_N(y) \cap G_N \neq \emptyset) \quad (91)$$

$$\chi_N^I(p, \sigma, \gamma; y) = E_{p, \sigma, \gamma}^I(|C_N(y)| \mathbb{1}\{C_N(y) \cap G_N = \emptyset\}). \quad (92)$$

That is, $\theta_N^I(p, \sigma, \gamma; y)$ is the probability that there is an open path from y to g . Also write

$$\theta_N^I(p, \sigma, \gamma) := \theta_N^I(p, \sigma, \gamma; 0) \quad \text{and} \quad \chi_N^I(p, \sigma, \gamma) := \chi_N^I(p, \sigma, \gamma; 0).$$

Then $\theta_N^I(p, \sigma, \gamma) = \theta_N^I(p, \sigma, \gamma; y)$ for all $y \in \mathbb{L}_0(N)$, and similarly for χ_N^I . We shall frequently simplify notation by leaving out the arguments when there is no risk of ambiguity, e.g. $\theta_N^I \equiv \theta_N^I(p, \sigma, \gamma)$. We have

$$\begin{aligned} \theta_N^I(p, \sigma, \gamma) &= 1 - \sum_{n=1}^{\infty} (1 - \gamma)^n P_{p, \sigma}^I(|C_N| = n) \quad \text{and} \\ \chi_N^I(p, \sigma, \gamma) &= \sum_{n=1}^{\infty} n (1 - \gamma)^n P_{p, \sigma}^I(|C_N| = n). \end{aligned}$$

We immediately obtain the following analogue of Equation (27):

$$\chi_N^I(p, \sigma, \gamma) = (1 - \gamma) \frac{\partial \theta_N^I}{\partial \gamma}. \quad (93)$$

The functions $\theta^I(p, \sigma, \gamma)$ and $\chi^I(p, \sigma, \gamma)$ are defined in the usual way for the infinite lattice \mathbb{L} with the ghost vertex g and edges \mathbb{E}_g , and with \mathbb{L}_0 as defined before:

$$\begin{aligned} \theta^I(p, \sigma, \gamma) &= \theta^I(p, \sigma, \gamma; 0) = P_{p, \sigma, \gamma}^I(|C| = \infty) \\ \chi^I(p, \sigma, \gamma) &= \chi^I(p, \sigma, \gamma; 0) = E_{p, \sigma, \gamma}^I(|C| \mathbb{1}\{C \cap G = \emptyset\}) \end{aligned}$$

where C is the cluster at the origin. Note that for $\gamma > 0$, $C \cap G$ is not empty with probability one when $|C| = \infty$.

The proof of the following lemma is similar to the proof for homogeneous percolation in appendix I of reference [17].

Lemma 5 *For all $\gamma \in (0, 1)$ and $p, \sigma \in (0, 1)$,*

$$\lim_{N \rightarrow \infty} \theta_N^I(p, \sigma, \gamma) = \theta^I(p, \sigma, \gamma),$$

and similarly,

$$\lim_{N \rightarrow \infty} \frac{\partial \theta_N^I}{\partial p} = \frac{\partial \theta^I}{\partial p}, \quad \lim_{N \rightarrow \infty} \frac{\partial \theta_N^I}{\partial \sigma} = \frac{\partial \theta^I}{\partial \sigma}, \quad \text{and} \quad \lim_{N \rightarrow \infty} \frac{\partial \theta_N^I}{\partial \gamma} = \frac{\partial \theta^I}{\partial \gamma}.$$

□

By equations (27) and (93) and lemma 5, we have

$$\lim_{N \rightarrow \infty} \chi_N^I(p, \sigma, \gamma) = \chi^I(p, \sigma, \gamma). \quad (94)$$

A.1 Uniform bounds for vertex-dependent functions

Let $P_{p,\gamma}^H$ and $E_{p,\gamma}^H$ be the probability measure and expectation for homogeneous percolation with a ghost field. Let $\chi_N^H(p, \gamma) = E_{p,\gamma}^H(|C_N| \mathbb{1}\{C_N \cap G_N = \emptyset\})$ be the corresponding susceptibility in $B(N)$. Also let $\chi_N^H(p) = \chi_N^H(p, 0)$.

Proving the differential inequalities requires the following useful lemma.

Lemma 6 *Assume $p \leq \sigma$ and $y \in B(N)$. Then*

- (a) $\theta_N^I(p, \sigma, \gamma; y) \leq \chi_N^H(p) \theta_N^I(p, \sigma, \gamma)$ for every $\gamma \in (0, 1)$;
- (b) $\chi_N^I(p, \sigma; y) \leq \chi_N^H(p) \chi_N^I(p, \sigma; 0)$ (where $\chi_N^I(p, \sigma; y) := \chi_N^I(p, \sigma, 0; y)$).

Proof Since $\chi_N^H(p) \geq 1$, it is only necessary to consider the case that $y \notin \mathbb{L}_0(N)$.

(a) Suppose that $C_N(y) \cap G_N \neq \emptyset$. Then there is an open path from y to a point of G_N . This path either uses no edge of $\mathbb{E}_0(N)$, or there exists an open self-avoiding path π from y to a vertex of G_N which passes through an edge of $\mathbb{E}_0(N)$. In the latter case, let z be the earliest point of π that is an endpoint of a bond in $\pi \cap \mathbb{L}_0(N)$. Then $z \in \mathbb{L}_0(N)$, and the part of π from y to z is disjoint from the part of the path from z to a vertex of G_N .

We formalize the above as follows. For given fixed y and for each $z \in \mathbb{L}_0(N)$, define the events

- \tilde{A}_N is the event that there is an open path from y to G_N in $\mathbb{E}(N) \setminus \mathbb{E}_0(N)$;
- $\tilde{D}_N(z)$ is the event that there is an open path from y to z in $\mathbb{E}(N) \setminus \mathbb{E}_0(N)$;
- $D_N^*(z)$ is the event that there is an open path from z to G_N in $B(N)$.

Observe that

$$P_{p,\sigma,\gamma}^I(\tilde{A}_N) = P_{p,\gamma}^H(\tilde{A}_N) \leq \theta_N^I(p, p, \gamma) \leq \theta_N^I(p, \sigma, \gamma) \quad (\text{since } p \leq \sigma). \quad (95)$$

Using standard percolation notation, $\tilde{D}_N(z) \circ D_N^*(z)$ is the event that $\tilde{D}_N(z)$ and $D_N^*(z)$ occur *disjointly*—that is, there exist two disjoint sets of open edges such that the first set guarantees occurrence of $\tilde{D}_N(z)$ and the second set guarantees occurrence of $D_N^*(z)$.

We then observe that

$$\{C_N(y) \cap G_N \neq \emptyset\} \subset \tilde{A}_N \cup \bigcup_{z \in \mathbb{L}_0(N)} (\tilde{D}_N(z) \circ D_N^*(z)) \quad (96)$$

since occurrence of the event $\{C_N(y) \cap G_N \neq \emptyset\}$ implies that either \tilde{A}_N occurs or $(\tilde{D}_N(z) \circ D_N^*(z))$ occurs for some $z \in \mathbb{L}_0(N)$.

From Equations (96) and (95) and the BK Inequality [17], we see that

$$\begin{aligned} P_{p,\sigma,\gamma}^I(C_N(y) \cap G_N \neq \emptyset) &\leq P_{p,\sigma,\gamma}^I(\tilde{A}_N) + \sum_{z \in \mathbb{L}_0(N)} P_{p,\sigma,\gamma}^I(\tilde{D}_N(z) \circ D_N^*(z)) \\ &\leq \theta_N^I(p, \sigma, \gamma) + \sum_{z \in \mathbb{L}_0(N)} P_{p,\sigma,\gamma}^I(\tilde{D}_N(z)) P_{p,\sigma,\gamma}^I(D_N^*(z)) \\ &\leq \theta_N^I(p, \sigma, \gamma) + \sum_{z \in \mathbb{L}_0(N)} P_p^H(z \in C_N(y)) P_{p,\sigma,\gamma}^I(C_N(z) \cap G_N \neq \emptyset) \\ &= \left(1 + \sum_{z \in \mathbb{L}_0(N)} P_p^H(z \in C_N(y))\right) \theta_N^I(p, \sigma, \gamma) \\ &\leq \chi_N^H(p) \theta_N^I(p, \sigma, \gamma). \end{aligned}$$

(b) Fix $y \in \mathbb{L}(N) \setminus \mathbb{L}_0(N)$. Let $\tilde{D}_N(z)$ be defined as in part (a). For $w \in B(N)$, let $\{y \leftrightarrow_N w\}$ denote the event that y and w are connected in $B(N)$ by a path of open edges.

Similarly to the proof of part (a), we see that for each $w \in B(N)$

$$\begin{aligned} P_{p,\sigma}^I(y \leftrightarrow_N w) &\leq P_{p,\sigma}^I(\tilde{D}_N(w)) + \sum_{z \in \mathbb{L}_0(N)} P_{p,\sigma}^I(\tilde{D}_N(z) \circ \{z \leftrightarrow_N w\}) \\ &\leq P_{p,\sigma}^I(\tilde{D}_N(w)) + \sum_{z \in \mathbb{L}_0(N)} P_{p,\sigma}^I(\tilde{D}_N(z)) P_{p,\sigma}^I(\{z \leftrightarrow_N w\}) \\ &\leq P_p^H(\{y \leftrightarrow_N w\}) + \sum_{z \in \mathbb{L}_0(N)} P_p^H(\{y \leftrightarrow_N z\}) P_{p,\sigma}^I(\{z \leftrightarrow_N w\}). \end{aligned} \quad (97)$$

Since $p \leq \sigma$, we have $\chi_N^H(p) \leq \chi_N^I(p, \sigma; 0) = \chi_N^I(p, \sigma; z)$ for every $z \in \mathbb{L}_0(N)$. Using this and summing equation (97) over w , we obtain

$$\begin{aligned} \chi_N^I(p, \sigma; y) &= \sum_{w \in B(N)} P_{p,\sigma}^I(y \leftrightarrow_N w) \\ &\leq \sum_{w \in B(N)} P_p^H(\{y \leftrightarrow_N w\}) + \sum_{w \in B(N)} \sum_{z \in \mathbb{L}_0(N)} P_p^H(\{y \leftrightarrow_N z\}) P_{p,\sigma}^I(\{z \leftrightarrow_N w\}) \\ &= \chi_N^H(p) + \sum_{z \in \mathbb{L}_0(N)} P_p^H(\{y \leftrightarrow_N z\}) \chi_N^I(p, \sigma; z) \\ &\leq \left(1 + \sum_{z \in \mathbb{L}_0(N)} P_p^H(\{y \leftrightarrow_N z\})\right) \chi_N^I(p, \sigma; 0) \leq \chi_N^H(p) \chi_N^I(p, \sigma; 0). \end{aligned}$$

This completes the proof.

A.2 The first differential inequality

The first differential inequality is defined in terms of $\theta_N^I \equiv \theta_N^I(p, \sigma, \gamma)$ as follows.

Lemma 7 For $p, \sigma, \gamma \in (0, 1)$, let $\mathbf{p} = (p, \sigma)$ and $\nabla \equiv \left(\frac{\partial}{\partial p}, \frac{\partial}{\partial \sigma}\right)$. Define $\mathbf{q} = \mathbf{1} - \mathbf{p} = (1 - p, 1 - \sigma)$. If $p \leq \sigma$, then we have

$$\mathbf{q} \cdot \nabla \theta_N^I \leq 2d \chi_N^H(p) (1 - \gamma) \theta_N^I \frac{\partial \theta_N^I}{\partial \gamma}.$$

Proof The proof is similar to the proof for homogeneous percolation (see for example reference [17]) and proceeds by applying Russo's formula to the event $\{C_N \cap G_N \neq \emptyset\}$, conditioned on G_N .

Let Γ be a realisation of G_N , i.e. a subset of vertices of $B(N)$. The event $A_N(\Gamma) = \{C_N \cap \Gamma \neq \emptyset\}$ is increasing. Hence by Russo's formula [17],

$$\frac{\partial}{\partial p} P_{p,\sigma}^I(A_N(\Gamma)) = \sum_{e \in \mathbb{E}(N) \setminus \mathbb{E}_0(N)} P_{p,\sigma}^I(e \text{ is pivotal for } A_N(\Gamma)), \quad (98)$$

$$\frac{\partial}{\partial \sigma} P_{p,\sigma}^I(A_N(\Gamma)) = \sum_{e \in \mathbb{E}_0(N)} P_{p,\sigma}^I(e \text{ is pivotal for } A_N(\Gamma)). \quad (99)$$

First consider equation (98). An edge $e = x \sim y$ is pivotal for $A_N(\Gamma)$ if and only if the following all occur in $\mathbb{E}(N) \setminus \{e\}$: (1) there is no open path from the origin

to Γ , (2) exactly one of x and y is joined to the origin by an open path, and (3) the other vertex is joined to a vertex of Γ by an open path. Hence,

$$\begin{aligned} & (1-p) \frac{\partial}{\partial p} P_{p,\sigma}^I(A_N(\Gamma)) \\ &= \sum_{e \in \mathbb{E}(N) \setminus \mathbb{E}_0(N)} P_{p,\sigma}^I(e \text{ is closed}) P_{p,\sigma}^I(e \text{ is pivotal for } A_N(\Gamma)) \\ &= \sum_{x \sim y \in \mathbb{E}(N) \setminus \mathbb{E}_0(N)} P_{p,\sigma}^I(x \in C_N, C_N \cap \Gamma = \emptyset, C_N(y) \cap \Gamma \neq \emptyset). \end{aligned}$$

where the last summation is over all ordered pairs (x, y) of vertices such that the (undirected) edge $x \sim y$ is in $\mathbb{E}(N) \setminus \mathbb{E}_0(N)$. Put $q = 1 - p$ and average the left hand side of the above over Γ :

$$\begin{aligned} & q \sum_{\Gamma} P_{p,\sigma,\gamma}^I(G_N = \Gamma) \frac{\partial}{\partial p} P_{p,\sigma}^I(A_N(\Gamma)) \\ &= q \frac{\partial}{\partial p} \left[\sum_{\Gamma} P_{p,\sigma,\gamma}^I(C_N \cap \Gamma \neq \emptyset) P_{p,\sigma,\gamma}^I(G_N = \Gamma) \right] \\ &= q \frac{\partial}{\partial p} P_{p,\sigma,\gamma}^I(C_N \cap G_N \neq \emptyset) \\ &= q \frac{\partial}{\partial p} \theta_N^I(p, \sigma, \gamma). \end{aligned}$$

Here it is important that the sum over Γ has a finite number of terms.

This shows that

$$q \frac{\partial}{\partial p} \theta_N^I = \sum_{x \sim y \in \mathbb{E}(N) \setminus \mathbb{E}_0(N)} P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset, C_N(y) \cap G_N \neq \emptyset).$$

Observe that C_N and $C_N(y)$ must be disjoint on the right hand side.

Exactly the same set of arguments applied to equation (99) gives (with $\tau = 1 - \sigma$)

$$\tau \frac{\partial}{\partial \sigma} \theta_N^I = \sum_{x \sim y \in \mathbb{E}_0(N)} P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset, C_N(y) \cap G_N \neq \emptyset). \quad (100)$$

Adding the last two equations together then produces

$$\mathbf{q} \cdot \nabla \theta_N^I = \sum_{x \sim y \in \mathbb{E}(N)} P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset, C_N(y) \cap G_N \neq \emptyset). \quad (101)$$

The right hand side of equation (101) must be bounded next. This is done by conditioning on the cluster at the origin. The last equation becomes

$$\begin{aligned} & \mathbf{q} \cdot \nabla \theta_N^I \\ &= \sum_{x \sim y} \left[\sum_{\Xi} P_{p,\sigma}^I(C_N = \Xi) P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset, C_N(y) \cap G_N \neq \emptyset \mid C_N = \Xi) \right] \end{aligned} \quad (102)$$

where the outer sum is over ordered pairs (x, y) such that $x \sim y \in \mathbb{E}(N)$, and the inner sum is over all connected graphs Ξ containing $\{0, x\}$ and not containing y .

Conditioned on $C_N = \Xi$, the events $C_N \cap G_N = \emptyset$ and $C_N(y) \cap G_N \neq \emptyset$ are independent (the first depends only on vertices of Ξ , and the second depends only on vertices and edges outside Ξ). Hence

$$\begin{aligned} & P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset, C_N(y) \cap G_N \neq \emptyset \mid C_N = \Xi) \\ &= P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset \mid C_N = \Xi) P_{p,\sigma,\gamma}^I(C_N(y) \cap G_N \neq \emptyset \mid C_N = \Xi). \end{aligned}$$

The condition $C_N = \Xi$ in the last factor restricts the set of possible open paths from y to a vertex in G_N (since $y \notin C_N$). Hence

$$P_{p,\sigma,\gamma}^I(C_N(y) \cap G_N \neq \emptyset \mid C_N = \Xi) \leq P_{p,\sigma,\gamma}^I(C_N(y) \cap G_N \neq \emptyset) = \theta_N^I(p, \sigma, \gamma; y).$$

This shows that

$$\begin{aligned} & \mathbf{q} \cdot \nabla \theta_N^I \\ & \leq \sum_{x \sim y} \left[\sum_{\Xi} P_{p,\sigma}^I(C_N = \Xi) P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset \mid C_N = \Xi) \theta_N^I(p, \sigma, \gamma; y) \right] \\ & = \sum_{x \sim y} \left[P_{p,\sigma,\gamma}^I(x \in C_N, y \notin C_N, C_N \cap G_N = \emptyset) \theta_N^I(p, \sigma, \gamma; y) \right] \\ & \leq \chi_N^H(p) \theta_N^I(p, \sigma, \gamma) \sum_{x \sim y} P_{p,\sigma,\gamma}^I(x \in C_N, y \notin C_N, C_N \cap G_N = \emptyset) \end{aligned} \quad (103)$$

where the final inequality comes from lemma 6(a). It remains to bound the last summation.

$$\begin{aligned} & \sum_{x \sim y} P_{p,\sigma,\gamma}^I(x \in C_N, y \notin C_N, C_N \cap G_N = \emptyset) \\ & \leq 2d \sum_x P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset) \\ & = 2d E_{p,\sigma,\gamma}^I(|C_N| \mathbb{1}\{C_N \cap G_N = \emptyset\}) \\ & = 2d \chi_N^I(p, \sigma, \gamma) = 2d(1 - \gamma) \frac{\partial \theta_N^I}{\partial \gamma} \end{aligned}$$

by equations (92) and (93). Putting this all together then gives the desired inequality.

A.3 The second differential inequality

The second differential inequality is the following (again writing θ_N^I for $\theta_N^I(p, \sigma, \gamma)$).

Lemma 8 For $p, \sigma, \gamma \in (0, 1)$ let $\mathbf{p} = (p, \sigma)$ and $\nabla \equiv \left(\frac{\partial}{\partial p}, \frac{\partial}{\partial \sigma} \right)$. If $p \leq \sigma$, then

$$\theta_N^I \leq \gamma \frac{\partial \theta_N^I}{\partial \gamma} + \left(\theta_N^I \right)^2 + \chi_N^H(p) \theta_N^I \left(\mathbf{p} \cdot \nabla \theta_N^I \right).$$

Proof Observe that

$$\begin{aligned} \theta_N^I(p, \sigma, \gamma) &= P_{p,\sigma,\gamma}^I(C_N \cap G_N \neq \emptyset) \\ &= P_{p,\sigma,\gamma}^I(|C_N \cap G_N| = 1) + P_{p,\sigma,\gamma}^I(|C_N \cap G_N| \geq 2). \end{aligned} \quad (104)$$

The first term in equation (104) can be calculated:

$$\begin{aligned} P_{p,\sigma,\gamma}^I(|C_N \cap G_N| = 1) &= \sum_{n=1}^{\infty} n \gamma (1 - \gamma)^{n-1} P_{p,\sigma,\gamma}^I(|C_N| = n) \\ &= \frac{\gamma \chi_N^I(p, \sigma, \gamma)}{1 - \gamma} \\ &= \gamma \frac{\partial \theta_N^I}{\partial \gamma} \end{aligned} \quad (105)$$

by equation (93).

It remains to bound the second term in (104). Define the event

$$A_x = \{x \in G_N \text{ or } x \text{ is joined to } G_N \text{ by an open path}\}.$$

Then $A_x \circ A_x$ is the event that there exist two distinct vertices v_1 and v_2 in G_N and two edge-disjoint open paths joining these vertices to x . If $x \in G_N$, then one of these paths may be the singleton x .

It follows that

$$\begin{aligned} P_{p,\sigma,\gamma}^I(|C_N \cap G_N| \geq 2) &= P_{p,\sigma,\gamma}^I(A_0 \circ A_0) \\ &+ P_{p,\sigma,\gamma}^I(|C_N \cap G_N| \geq 2, \text{ and } A_0 \circ A_0 \text{ does not occur}). \end{aligned} \quad (106)$$

By the BK inequality,

$$P_{p,\sigma,\gamma}^I(A_0 \circ A_0) \leq \left(P_{p,\sigma,\gamma}^I(A_0)\right)^2 = \left(\theta_N^I(p, \sigma, \gamma)\right)^2. \quad (107)$$

The remaining term is the probability that $|C_N \cap G_N| \geq 2$ but there do not exist two edge-disjoint paths from the origin to distinct vertices in G_N . If this occurs, then there exists an edge $x \sim y$ in $\mathbb{E}(N)$ with the following properties:

- $x \sim y$ is open;
- If $x \sim y$ is deleted in $\mathbb{L}(N)$, then three events occur:
 1. there is no open path from the origin to a vertex of G_N ;
 2. x is joined to the origin by an open path;
 3. the event $A_y \circ A_y$ occurs.

The probability that a particular edge $x \sim y$ has these properties is

$$pq^{-1} P_{p,\sigma,\gamma}^I(x \sim y \text{ is closed}, x \in C_N, C_N \cap G_N = \emptyset, A_y \circ A_y)$$

if $x \sim y \in \mathbb{E}(N) \setminus \mathbb{E}_0(N)$; if $x \sim y \in \mathbb{E}_0(N)$, then we get the above expression with $\sigma(1 - \sigma)^{-1}$ instead of pq^{-1} . Therefore we obtain the bound

$$P_{p,\sigma,\gamma}^I(|C_N \cap G_N| \geq 2, \text{ and } A_0 \circ A_0 \text{ does not occur}) \leq S_1 + S_2,$$

where

$$S_1 = pq^{-1} \sum_{x \sim y \in \mathbb{E}(N) \setminus \mathbb{E}_0(N)} P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset, A_y \circ A_y) \quad (108)$$

$$\text{and } S_2 = \sigma\tau^{-1} \sum_{x \sim y \in \mathbb{E}_0(N)} P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset, A_y \circ A_y), \quad (109)$$

writing $\tau = 1 - \sigma$.

Consider a summand from equation (108) and (109) conditioned on $C_N = \Xi$, with $x \in \Xi$ and $y \notin \Xi$. Using conditional independence of the events A_y and $\{C_N \cap G_N = \emptyset\}$, and the BK inequality, we obtain

$$\begin{aligned} P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset, A_y \circ A_y \mid C_N = \Xi) &= P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset \mid C_N = \Xi) P_{p,\sigma,\gamma}^I(A_y \circ A_y \mid C_N = \Xi) \\ &\leq P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset \mid C_N = \Xi) \left(P_{p,\sigma,\gamma}^I(A_y \mid C_N = \Xi)\right)^2 \\ &\leq P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset \mid C_N = \Xi) P_{p,\sigma,\gamma}^I(A_y \mid C_N = \Xi) P_{p,\sigma,\gamma}^I(A_y) \\ &= P_{p,\sigma,\gamma}^I(C_N \cap G_N = \emptyset, A_y \mid C_N = \Xi) \theta_N^I(p, \sigma, \gamma; y). \end{aligned}$$

Substitute this into equation (108) and average over Ξ . This gives the upper bound

$$S_1 \leq pq^{-1} \sum_{x \sim y \in \mathbb{E}(N) \setminus \mathbb{E}_0(N)} P_{p,\sigma,\gamma}^I(x \in C_N, C_N \cap G_N = \emptyset, A_y) \theta_N^I(p, \sigma, \gamma; y).$$

Next, by equation (100) and lemma 6(a), we obtain (with $\theta_N^I(p, \sigma, \gamma) \equiv \theta_N^I$)

$$S_1 \leq pq^{-1} \left(q \frac{\partial \theta_N^I}{\partial p} \right) \chi_N^H(p) \theta_N^I.$$

The analogous bound for equation (109) is

$$S_2 \leq \sigma \tau^{-1} \left(\tau \frac{\partial \theta_N^I}{\partial \sigma} \right) \chi_N^H(p) \theta_N^I.$$

Hence,

$$\begin{aligned} P_{p,\sigma,\gamma}^I (|C_N \cap G_N| \geq 2, \text{ and } A_0 \circ A_0 \text{ does not occur}) \\ \leq \left(p \frac{\partial \theta_N^I}{\partial p} + \sigma \frac{\partial \theta_N^I}{\partial \sigma} \right) \chi_N^H(p) \theta_N^I. \end{aligned}$$

Putting this together with equations (104), (106), (105) and (107) completes the proof of the desired inequality.

A.4 The final differential inequalities

To complete the proof of the two differential inequalities (29) and (30), we take the $N \rightarrow \infty$ limit in lemmas 7 and 8. Using lemma 5 and equation (94), the result is the following theorem.

Theorem 8 For $p, \sigma, \gamma \in (0, 1)$, write $\mathbf{p} = (p, \sigma)$, $\nabla \equiv \left(\frac{\partial}{\partial p}, \frac{\partial}{\partial \sigma} \right)$, $\mathbf{q} = \mathbf{1} - \mathbf{p} = (1-p, 1-\sigma)$, and $\theta^I \equiv \theta^I(p, \sigma, \gamma)$. If $p \leq \sigma$, then

$$\begin{aligned} \mathbf{q} \cdot \nabla \theta^I &\leq 2d(1-\gamma) \chi^H(p) \theta^I \frac{\partial \theta^I}{\partial \gamma} \quad \text{and} \\ \theta^I &\leq \gamma \frac{\partial \theta^I}{\partial \gamma} + \left(\theta^I \right)^2 + \chi^H(p) \theta^I \left(\mathbf{p} \cdot \nabla \theta^I \right). \end{aligned}$$

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